

GLOBAL WELL-POSEDNESS OF THE COMPRESSIBLE BIPOLAR EULER-MAXWELL SYSTEM IN \mathbb{R}^3

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ABSTRACT. We first construct the global unique solution by assuming that the initial data is small in the H^3 norm but its higher order derivatives could be large. If further the initial data belongs to \dot{H}^{-s} ($0 \leq s < 3/2$) or $\dot{B}_{2,\infty}^{-s}$ ($0 < s \leq 3/2$), we obtain the various decay rates of the solution and its higher order derivatives. As an immediate byproduct, the L^p - L^2 ($1 \leq p \leq 2$) type of the decay rates follow without requiring the smallness for L^p norm of initial data. In particular, the decay rate for the difference of densities could reach to $(1+t)^{-\frac{13}{4}}$ in L^2 norm.

1. INTRODUCTION

We consider the compressible isentropic bipolar Euler-Maxwell system in three space dimensions [1, 17, 21]

$$\begin{cases} \partial_t \tilde{n}_\pm + \operatorname{div}(\tilde{n}_\pm \tilde{u}_\pm) = 0, \\ \partial_t(\tilde{n}_\pm \tilde{u}_\pm) + \operatorname{div}(\tilde{n}_\pm \tilde{u}_\pm \otimes \tilde{u}_\pm) + \nabla p_\pm(\tilde{n}_\pm) = \pm \tilde{n}_\pm(\tilde{E} + \varepsilon \tilde{u}_\pm \times \tilde{B}) - \frac{1}{\tau_\pm} \tilde{n}_\pm \tilde{u}_\pm, \\ \varepsilon \lambda^2 \partial_t \tilde{E} - \nabla \times \tilde{B} = \varepsilon(\tilde{n}_- \tilde{u}_- - \tilde{n}_+ \tilde{u}_+), \\ \varepsilon \partial_t \tilde{B} + \nabla \times \tilde{E} = 0, \\ \lambda^2 \operatorname{div} \tilde{E} = \tilde{n}_+ - \tilde{n}_-, \quad \operatorname{div} \tilde{B} = 0, \\ (\tilde{n}_\pm, \tilde{u}_\pm, \tilde{E}, \tilde{B})|_{t=0} = (\tilde{n}_{\pm 0}, \tilde{u}_{\pm 0}, \tilde{E}_0, \tilde{B}_0). \end{cases} \quad (1.1)$$

Here the unknown functions are the charged density \tilde{n}_\pm , the velocity \tilde{u}_\pm , the electric field \tilde{E} and the magnetic field \tilde{B} , with the subscripts $+$ and $-$ representing ion and electron respectively. We assume the pressure $p_\pm(\tilde{n}_\pm) = A_\pm \tilde{n}_\pm^\gamma$ with constants $A_\pm > 0$ and $\gamma \geq 1$ the adiabatic exponent. $1/\tau_\pm > 0$ are the velocity relaxation time of ions and electrons respectively. $\lambda > 0$ is the Debye length, and $\varepsilon = 1/c$ with c the speed of light.

Although its significance in plasma physics and semiconductor physics, there are merely few mathematical results about the compressible Euler-Maxwell system since its complexity in mathematics. For the unipolar case: Chen, Jerome and Wang [2] showed the global existence of entropy weak solutions to the initial-boundary value problem for arbitrarily large initial data in $L^\infty(\mathbb{R})$; Guo and Tahvildar-Zadeh [11] showed a blow-up criterion for spherically symmetric Euler-Maxwell system; Recently, there are some results on the global existence and the asymptotic behavior of smooth solutions with small amplitudes, see Tan et al. [24], Duan [3], Ueda and Kawashima [27], Ueda et al. [28]; For the asymptotic limits that derive simplified models starting from the Euler-Maxwell system, we refer to [13, 20, 31] for the relaxation limit, [31] for the non-relativistic limit, [18, 19] for the quasi-neutral limit, [25, 26] for WKB asymptotics and the references therein. For the bipolar case: Duan et al. [4] showed the global existence and time-decay rates of solutions near constant steady states with the vanishing electromagnetic field; Xu et al. [32] studied the well-posedness in critical Besov spaces. Since the unipolar or bipolar Euler-Maxwell system is a symmetrizable hyperbolic system, the Cauchy problem in \mathbb{R}^3 has a local unique smooth solution when the initial data is smooth, see Kato [15] and Jerome [14] for instance. Besides, we can refer to [5, 29] for the non-isentropic case.

In this paper, we will derive a refined global existence of smooth solutions near the constant equilibrium $(n_\infty, n_\infty, 0, 0, 0, B_\infty)$ to the compressible isentropic bipolar Euler-Maxwell system and show some various time decay rates of the solution as well as its spatial derivatives of any order. Because of the complexities and some new difficulties, we will study the compressible non-isentropic bipolar Euler-Maxwell system in the future work. We should notice that the relaxation term of the velocity plays an important role in the whole paper. The non-relaxation case is much more difficult, we refer to [6, 8, 10] for such a case. For the

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compressible unipolar Euler-Maxwell system [24], we do not need that the initial electron density belongs to negative Sobolev spaces \dot{H}^{-s} or negative Besov spaces $\dot{B}_{2,\infty}^{-s}$ when deriving the optimal decay rates of solutions. However, in Theorem 1.2 the initial total densities n_{10} must belong to \dot{H}^{-s} or $\dot{B}_{2,\infty}^{-s}$ since the cancelation between two carriers. In fact, in Theorem 1.2 the assumption for the initial difference of densities n_{20} could be deleted given [24]. Compared with [24], there are two major difficulties except the computational complexity. First of all, the bipolar system (1.1) could be reformulated equivalently as the damped Euler equations coupled with the one-fluid Euler-Maxwell equations (1.5). Then, the total densities n_1 in the damped Euler equations is degenerately dissipative because of the cancelation between two carriers. It is difficult to close the energy estimates since the degenerate dissipation of n_1 . We manage to obtain the effective energy estimates by dealing carefully with these terms involved with n_1 in the proofs of Lemma 2.8 and Lemma 2.9. The other difficulty is caused by the nonlinear function $f(\frac{n_1 \pm n_2}{2})$. Since n_1 and n_2 have different dissipative structures, we must be careful about the function $f(\frac{n_1 \pm n_2}{2})$. Here we overcome such an obstacle by some detailed calculi. Without loss of generality, we take all the physical constants $\tau_{\pm}, \varepsilon, \lambda, A_{\pm}, n_{\infty}$ in (1.1) to be one.

We define

$$\begin{cases} n_{\pm}(x, t) = \frac{2}{\gamma-1} \left\{ \left[\tilde{n}_{\pm}\left(x, \frac{t}{\sqrt{\gamma}}\right) \right]^{\frac{\gamma-1}{2}} - 1 \right\}, & u_{\pm}(x, t) = \frac{1}{\sqrt{\gamma}} \tilde{u}_{\pm}\left(x, \frac{t}{\sqrt{\gamma}}\right), \\ E(x, t) = \frac{1}{\sqrt{\gamma}} \tilde{E}\left(x, \frac{t}{\sqrt{\gamma}}\right), & B(x, t) = \frac{1}{\sqrt{\gamma}} \tilde{B}\left(x, \frac{t}{\sqrt{\gamma}}\right) - B_{\infty}. \end{cases} \quad (1.2)$$

Then the Euler-Maxwell system (1.1) is reformulated equivalently as

$$\begin{cases} \partial_t n_{\pm} + \operatorname{div} u_{\pm} = -u_{\pm} \cdot \nabla n_{\pm} - \mu n_{\pm} \operatorname{div} u_{\pm}, \\ \partial_t u_{\pm} + \nu u_{\pm} \mp u_{\pm} \times B_{\infty} + \nabla n_{\pm} \mp \nu E = -u_{\pm} \cdot \nabla u_{\pm} - \mu n_{\pm} \nabla n_{\pm} \pm u_{\pm} \times B, \\ \partial_t E - \nu \nabla \times B + \nu(u_+ - u_-) = \nu(f(n_-)u_- - f(n_+)u_+), \\ \partial_t B + \nu \nabla \times E = 0, \\ \operatorname{div} E = \nu(f(n_+) - f(n_-)), \quad \operatorname{div} B = 0, \\ (n_{\pm}, u_{\pm}, E, B)|_{t=0} = (n_{\pm 0}, u_{\pm 0}, E_0, B_0). \end{cases}$$

Here $\mu := \frac{\gamma-1}{2}$, $\nu := \frac{1}{\sqrt{\gamma}}$ and the nonlinear function $f(n_{\pm})$ is defined by

$$f(n_{\pm}) := \left(1 + \frac{\gamma-1}{2} n_{\pm}\right)^{\frac{2}{\gamma-1}} - 1. \quad (1.3)$$

In fact, we have assumed $\gamma > 1$ in (1.2). If $\gamma = 1$, we instead define $n_{\pm} := \ln \tilde{n}_{\pm}$.

Let

$$n_1 = n_+ + n_-, \quad n_2 = n_+ - n_-, \quad u_1 = u_+ + u_-, \quad u_2 = u_+ - u_-,$$

that is

$$n_+ = \frac{n_1 + n_2}{2}, \quad n_- = \frac{n_1 - n_2}{2}, \quad u_+ = \frac{u_1 + u_2}{2}, \quad u_- = \frac{u_1 - u_2}{2}. \quad (1.4)$$

Then $U := (n_1, n_2, u_1, u_2, E, B)$ satisfies

$$\begin{cases} \partial_t n_1 + \operatorname{div} u_1 = g_1, \\ \partial_t u_1 + \nu u_1 - u_2 \times B_{\infty} + \nabla n_1 = g_2 + u_2 \times B, \\ \partial_t n_2 + \operatorname{div} u_2 = g_3, \\ \partial_t u_2 + \nu u_2 - u_1 \times B_{\infty} + \nabla n_2 - 2\nu E = g_4 + u_1 \times B, \\ \partial_t E - \nu \nabla \times B + \nu u_2 = g_5, \\ \partial_t B + \nu \nabla \times E = 0, \\ \operatorname{div} E = \nu \left(f\left(\frac{n_1 + n_2}{2}\right) - f\left(\frac{n_1 - n_2}{2}\right) \right), \quad \operatorname{div} B = 0, \end{cases} \quad (1.5)$$

with initial data

$$U|_{t=0} = U_0 := (n_{10}, n_{20}, u_{10}, u_{20}, E_0, B_0).$$

Here

$$\begin{aligned}
g_1 &= -\frac{1}{2}(u_1 \cdot \nabla n_1 + u_2 \cdot \nabla n_2) - \frac{\mu}{2}(n_1 \operatorname{div} u_1 + n_2 \operatorname{div} u_2), \\
g_2 &= -\frac{1}{2}(u_1 \cdot \nabla u_1 + u_2 \cdot \nabla u_2) - \frac{\mu}{2}(n_1 \nabla n_1 + n_2 \nabla n_2), \\
g_3 &= -\frac{1}{2}(u_1 \cdot \nabla n_2 + u_2 \cdot \nabla n_1) - \frac{\mu}{2}(n_1 \operatorname{div} u_2 + n_2 \operatorname{div} u_1), \\
g_4 &= -\frac{1}{2}(u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1) - \frac{\mu}{2}(n_1 \nabla n_2 + n_2 \nabla n_1), \\
g_5 &= \nu \left(f \left(\frac{n_1 - n_2}{2} \right) \frac{u_1 - u_2}{2} - f \left(\frac{n_1 + n_2}{2} \right) \frac{u_1 + u_2}{2} \right).
\end{aligned} \tag{1.6}$$

Notations: In this paper, we use $H^s(\mathbb{R}^3)$, $s \in \mathbb{R}$ to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^s}$ and $L^p(\mathbb{R}^3)$, $1 \leq p \leq \infty$ to denote the usual L^p spaces with norm $\|\cdot\|_{L^p}$. ∇^ℓ with an integer $\ell \geq 0$ stands for the usual any spatial derivatives of order ℓ . When $\ell < 0$ or ℓ is not a positive integer, ∇^ℓ stands for Λ^ℓ defined by $\Lambda^\ell f := \mathcal{F}^{-1}(|\xi|^\ell \mathcal{F}f)$, where \mathcal{F} is the usual Fourier transform operator and \mathcal{F}^{-1} is its inverse. We use $\dot{H}^s(\mathbb{R}^3)$, $s \in \mathbb{R}$ to denote the homogeneous Sobolev spaces on \mathbb{R}^3 with norm $\|\cdot\|_{\dot{H}^s}$ defined by $\|f\|_{\dot{H}^s} := \|\Lambda^s f\|_{L^2}$. We then recall the homogeneous Besov spaces. Let $\phi \in C_c^\infty(\mathbb{R}^3_\xi)$ be such that $\phi(\xi) = 1$ when $|\xi| \leq 1$ and $\phi(\xi) = 0$ when $|\xi| \geq 2$. Let $\varphi(\xi) = \phi(\xi) - \phi(2\xi)$ and $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ for $j \in \mathbb{Z}$. Then by the construction, $\sum_{j \in \mathbb{Z}} \varphi_j(\xi) = 1$ if $\xi \neq 0$. We define $\dot{\Delta}_j f := \mathcal{F}^{-1}(\varphi_j) * f$, then for $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, we define the homogeneous Besov spaces $\dot{B}_{p,r}^s(\mathbb{R}^3)$ with norm $\|\cdot\|_{\dot{B}_{p,r}^s}$ defined by

$$\|f\|_{\dot{B}_{p,r}^s} := \left(\sum_{j \in \mathbb{Z}} 2^{rsj} \|\dot{\Delta}_j f\|_{L^p}^r \right)^{\frac{1}{r}}.$$

Particularly, if $r = \infty$, then

$$\|f\|_{\dot{B}_{p,\infty}^s} := \sup_{j \in \mathbb{Z}} 2^{sj} \|\dot{\Delta}_j f\|_{L^p}.$$

Throughout this paper we let C denote some positive (generally large) universal constants and λ denote some positive (generally small) universal constants. They *do not* depend on either k or N ; otherwise, we will denote them by C_k , C_N , etc. We will use $a \lesssim b$ if $a \leq Cb$, and $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$. We use C_0 to denote the constants depending on the initial data and k, N, s . For simplicity, we write $\|(A, B)\|_X := \|A\|_X + \|B\|_X$ and $\int f := \int_{\mathbb{R}^3} f dx$. $(*) \times \varepsilon + (**)$ denote that multiplying $(*)$ by a sufficiently small but fixed factor ε and then adding it to $(**)$.

For $N \geq 3$, we define the energy functional by

$$\mathcal{E}_N(t) := \sum_{l=0}^N \|\nabla^l U\|_{L^2}^2$$

and the corresponding dissipation rate by

$$\mathcal{D}_N(t) := \sum_{l=1}^N \|\nabla^l n_1\|_{L^2}^2 + \sum_{l=0}^N \|\nabla^l(n_2, u_1, u_2)\|_{L^2}^2 + \sum_{l=0}^{N-1} \|\nabla^l E\|_{L^2}^2 + \sum_{l=1}^{N-1} \|\nabla^l B\|_{L^2}^2.$$

Our first main result about the global unique solution to the system (1.5) is stated as follows.

Theorem 1.1. *Assume the initial data satisfy the compatible conditions*

$$\operatorname{div} E_0 = \nu \left(f \left(\frac{n_{10} + n_{20}}{2} \right) - f \left(\frac{n_{10} - n_{20}}{2} \right) \right), \quad \operatorname{div} B_0 = 0.$$

There exists a sufficiently small $\delta_0 > 0$ such that if $\mathcal{E}_3(0) \leq \delta_0$, then there exists a unique global solution $U(t)$ to the Euler-Maxwell system (1.5) satisfying

$$\sup_{0 \leq t \leq \infty} \mathcal{E}_3(t) + \int_0^\infty \mathcal{D}_3(\tau) d\tau \leq C \mathcal{E}_3(0). \tag{1.7}$$

Furthermore, if $\mathcal{E}_N(0) < +\infty$ for any $N \geq 3$, there exists an increasing continuous function $P_N(\cdot)$ with $P_N(0) = 0$ such that the unique solution satisfies

$$\sup_{0 \leq t \leq \infty} \mathcal{E}_N(t) + \int_0^\infty \mathcal{D}_N(\tau) d\tau \leq P_N(\mathcal{E}_N(0)). \tag{1.8}$$

In the proof of Theorem 1.1, the major difficulties are caused by the degenerate dissipation for the total densities and the regularity-loss of the electromagnetic field. We will do the refined energy estimates stated in Lemma 2.8–2.9, which allow us to deduce

$$\frac{d}{dt}\mathcal{E}_3 + \mathcal{D}_3 \lesssim \sqrt{\mathcal{E}_3}\mathcal{D}_3$$

and for $N \geq 4$,

$$\frac{d}{dt}\mathcal{E}_N + \mathcal{D}_N \leq C_N \mathcal{D}_{N-1} \mathcal{E}_N.$$

Then Theorem 1.1 follows in the fashion of [9, 30, 24].

Our second main result is on some various decay rates of the solution to the system (1.5) by making the much stronger assumption on the initial data.

Theorem 1.2. *Assume that $U(t)$ is the solution to the Euler-Maxwell system (1.5) constructed in Theorem 1.1 with $N \geq 5$. There exists a sufficiently small $\delta_0 = \delta_0(N)$ such that if $\mathcal{E}_N(0) \leq \delta_0$, and assuming that $U_0 \in \dot{H}^{-s}$ for some $s \in [0, 3/2]$ or $U_0 \in \dot{B}_{2,\infty}^{-s}$ for some $s \in (0, 3/2]$, then we have*

$$\|U(t)\|_{\dot{H}^{-s}} \leq C_0 \quad (1.9)$$

or

$$\|U(t)\|_{\dot{B}_{2,\infty}^{-s}} \leq C_0. \quad (1.10)$$

Moreover, for any fixed integer $k \geq 0$, if $N \geq 2k + 2 + s$, then

$$\|\nabla^k U(t)\|_{L^2} \leq C_0(1+t)^{-\frac{k+s}{2}}. \quad (1.11)$$

Furthermore, for any fixed integer $k \geq 0$, if $N \geq 2k + 4 + s$, then

$$\|\nabla^k(n_2, u_1, u_2, E)(t)\|_{L^2} \leq C_0(1+t)^{-\frac{k+1+s}{2}}; \quad (1.12)$$

if $N \geq 2k + 6 + s$, then

$$\|\nabla^k n_2(t)\|_{L^2} \leq C_0(1+t)^{-\frac{k+2+s}{2}}; \quad (1.13)$$

if $N \geq 2k + 12 + s$ and $B_\infty = 0$, then

$$\|\nabla^k(n_2, \operatorname{div} u_2)(t)\|_{L^2} \leq C_0(1+t)^{-(\frac{k}{2} + \frac{7}{4} + s)}. \quad (1.14)$$

In the proof of Theorem 1.2, we mainly use the regularity interpolation method developed in Strain and Guo [23], Guo and Wang [12] and Sohinger and Strain [22]. To prove the optimal decay rate of the dissipative equations in the whole space, Guo and Wang [12] developed a general energy method of using a family of scaled energy estimates with minimum derivative counts and interpolations among them. However, this method can not be applied directly to the compressible bipolar Euler-Maxwell system which is of regularity-loss. To overcome this obstacle caused by the regularity-loss of the electromagnetic field, we deduce from Lemma 2.8–2.9 that

$$\frac{d}{dt}\mathcal{E}_k^{k+2} + \mathcal{D}_k^{k+2} \leq C_k \|(n_2, u_1, u_2)\|_{L^\infty} \|\nabla^{k+2}(n_1, n_2, u_1, u_2)\|_{L^2} \|\nabla^{k+2}(E, B)\|_{L^2},$$

where \mathcal{E}_k^{k+2} and \mathcal{D}_k^{k+2} with minimum derivative counts are defined by (3.5) and (3.6) respectively. Then combining the methods of [12, 22] and a trick of Strain and Guo [23] to treat the electromagnetic field, we manage to conclude the decay rate (1.11). If in view of the whole solution, the decay rate (1.11) can be regarded as be optimal. The higher decay rates (1.12)–(1.14) follow by revisiting the equations carefully. In particular, we will use a bootstrap argument to derive (1.14).

By Theorem 1.2 and Lemma 2.4–2.5, we have the following corollary of the usual L^p – L^2 type of the decay results:

Corollary 1.3. *Under the assumptions of Theorem 1.2 except that we replace the \dot{H}^{-s} or $\dot{B}_{2,\infty}^{-s}$ assumption by that $U_0 \in L^p$ for some $p \in [1, 2]$, then for any fixed integer $k \geq 0$, if $N \geq 2k + 2 + s_p$, then*

$$\|\nabla^k U(t)\|_{L^2} \leq C_0(1+t)^{-\frac{k+s_p}{2}}.$$

Here the number $s_p := 3\left(\frac{1}{p} - \frac{1}{2}\right)$.

Furthermore, for any fixed integer $k \geq 0$, if $N \geq 2k + 4 + s_p$, then

$$\|\nabla^k(n_2, u_1, u_2, E)(t)\|_{L^2} \leq C_0(1+t)^{-\frac{k+1+s_p}{2}};$$

if $N \geq 2k + 6 + s_p$, then

$$\|\nabla^k n_2(t)\|_{L^2} \leq C_0(1+t)^{-\frac{k+2+s_p}{2}};$$

if $N \geq 2k + 12 + s_p$ and $B_\infty = 0$, then

$$\|\nabla^k(n_2, \operatorname{div} u_2)(t)\|_{L^2} \leq C_0(1+t)^{-(\frac{k}{2} + \frac{7}{4} + s_p)}. \quad (1.15)$$

The followings are several remarks on Theorem 1.1–1.2 and Corollary 1.3.

Remark 1.4. In Theorem 1.1, we only assume that the initial data is small in the H^3 norm but the higher order derivatives could be large. Notice that in Theorem 1.2 the \dot{H}^{-s} and $\dot{B}_{2,\infty}^{-s}$ norms of the solution are preserved along the time evolution, however, in Corollary 1.3 it is difficult to show that the L^p norm of the solution can be preserved. Note that the L^2 decay rate of the higher order spatial derivatives of the solution is obtained. Then the general optimal L^q ($2 \leq q \leq \infty$) decay rates of the solution follow by the Sobolev interpolation.

Remark 1.5. In Theorem 1.2, the space \dot{H}^{-s} or $\dot{B}_{2,\infty}^{-s}$ was introduced there to enhance the decay rates. By the usual embedding theorem, we know that for $p \in (1, 2]$, $L^p \subset \dot{H}^{-s}$ with $s = 3(\frac{1}{p} - \frac{1}{2}) \in [0, 3/2]$. Meantime, we note that the endpoint embedding $L^1 \subset \dot{B}_{2,\infty}^{-\frac{3}{2}}$ holds. Hence the L^p – L^2 ($1 \leq p \leq 2$) type of the optimal decay results follows as a corollary.

Remark 1.6. We remark that Corollary 1.3 not only provides an alternative approach to derive the L^p – L^2 type of the optimal decay results but also improves the previous results of the L^p – L^2 approach in Duan et al. [4]. In Duan et al. [4], assuming that $B_\infty = 0$ and $\|U_0\|_{L^1}$ is sufficiently small, by combining the energy method and the linear decay analysis, Duan proved that

$$\|n_2(t)\|_{L^2} \leq C_0(1+t)^{-\frac{5}{2}}, \quad \|(u_1, u_2, E)(t)\|_{L^2} \leq C_0(1+t)^{-\frac{5}{4}} \text{ and } \|(n_1, B)(t)\|_{L^2} \leq C_0(1+t)^{-\frac{3}{4}}.$$

Notice that for $p = 1$, our decay rate of $n_2(t)$ is $(1+t)^{-13/4}$ in (1.15).

The rest of our paper is structured as follows. In section 2, we establish the refined energy estimates for the solution and derive the negative Sobolev and Besov estimates. Theorem 1.1 and Theorem 1.2 are proved in section 3.

2. NONLINEAR ENERGY ESTIMATES

In this section, we will do the a priori estimate by assuming that $\|n_\pm(t)\|_{H^3} \leq \delta \ll 1$. Recall the expression (1.3) of $f(n_\pm)$ and (1.4). Then by Taylor's formula and Sobolev's inequality, we have

$$f\left(\frac{n_1 \pm n_2}{2}\right) \sim \frac{n_1 \pm n_2}{2} \text{ and } \left|f^{(k)}\left(\frac{n_1 \pm n_2}{2}\right)\right| \leq C_k \text{ for any } k \geq 1. \quad (2.1)$$

2.1. Preliminary. In this subsection, we collect some analytic tools used later in this paper.

Lemma 2.1. Let $2 \leq p \leq +\infty$ and $\alpha, m, \ell \geq 0$. Then we have

$$\|\nabla^\alpha f\|_{L^p} \leq C_p \|\nabla^m f\|_{L^2}^{1-\theta} \|\nabla^\ell f\|_{L^2}^\theta.$$

Here $0 \leq \theta \leq 1$ (if $p = +\infty$, then we require that $0 < \theta < 1$) and α satisfies

$$\alpha + 3\left(\frac{1}{2} - \frac{1}{p}\right) = m(1 - \theta) + \ell\theta.$$

Proof. For the case $2 \leq p < +\infty$, we refer to Lemma A.1 in [12]; for the case $p = +\infty$, we refer to Exercise 6.1.2 in [7] (pp. 421). \square

Lemma 2.2. For any integer $k \geq 0$, we have

$$\|\nabla^k f(n)\|_{L^\infty} \leq C_k \|\nabla^{k+1} n\|_{L^2}^{1/2} \|\nabla^{k+2} n\|_{L^2}^{1/2}, \quad (2.2)$$

and

$$\|\nabla^k f(n)\|_{L^2} \leq C_k \|\nabla^k n\|_{L^2}.$$

Proof. See Lemma 2.2 in [24]. \square

We recall the following commutator estimate:

Lemma 2.3. Let $k \geq 1$ be an integer and define the commutator

$$[\nabla^k, g]h = \nabla^k(gh) - g\nabla^k h.$$

Then we have

$$\|[\nabla^k, g]h\|_{L^2} \leq C_k (\|\nabla g\|_{L^\infty} \|\nabla^{k-1} h\|_{L^2} + \|\nabla^k g\|_{L^2} \|h\|_{L^\infty}),$$

and

$$\|\nabla^k(gh)\|_{L^2} \leq C_k (\|g\|_{L^\infty} \|\nabla^k h\|_{L^2} + \|\nabla^k g\|_{L^2} \|h\|_{L^\infty}).$$

Proof. It can be proved by using Lemma 2.1, see Lemma 3.4 in [16] (pp. 98) for instance. \square

Notice that when using the commutator estimate in this paper, we usually will not consider the case that $k = 0$ since it is trivial.

We have the L^p embeddings:

Lemma 2.4. *Let $0 \leq s < 3/2$, $1 < p \leq 2$ with $1/2 + s/3 = 1/p$, then*

$$\|f\|_{\dot{H}^{-s}} \lesssim \|f\|_{L^p}.$$

Proof. It follows from the Hardy-Littlewood-Sobolev theorem, see [7]. \square

Lemma 2.5. *Let $0 < s \leq 3/2$, $1 \leq p < 2$ with $1/2 + s/3 = 1/p$, then*

$$\|f\|_{\dot{B}_{2,\infty}^{-s}} \lesssim \|f\|_{L^p}.$$

Proof. See Lemma 4.6 in [22]. \square

It is important to use the following special interpolation estimates:

Lemma 2.6. *Let $s \geq 0$ and $\ell \geq 0$, then we have*

$$\|\nabla^\ell f\|_{L^2} \leq \|\nabla^{\ell+1} f\|_{L^2}^{1-\theta} \|f\|_{\dot{H}^{-s}}^\theta, \text{ where } \theta = \frac{1}{\ell+1+s}.$$

Proof. It follows directly by the Parseval theorem and Hölder's inequality. \square

Lemma 2.7. *Let $s > 0$ and $\ell \geq 0$, then we have*

$$\|\nabla^\ell f\|_{L^2} \leq \|\nabla^{\ell+1} f\|_{L^2}^{1-\theta} \|f\|_{\dot{B}_{2,\infty}^{-s}}^\theta, \text{ where } \theta = \frac{1}{\ell+1+s}.$$

Proof. See Lemma 4.5 in [22]. \square

2.2. Energy estimates. In this subsection, we will derive the basic energy estimates for the solution to the Euler-Maxwell system (1.5). We begin with the standard energy estimates.

Lemma 2.8. *For any integer $k \geq 0$, we have*

$$\begin{aligned} & \frac{d}{dt} \sum_{l=k}^{k+2} \|\nabla^l U\|_{L^2}^2 + \lambda \sum_{l=k}^{k+2} \|\nabla^l(u_1, u_2)\|_{L^2}^2 \\ & \lesssim C_k F \left(\sum_{l=k+1}^{k+2} \|\nabla^l n_1\|_{L^2}^2 + \sum_{l=k}^{k+2} \|\nabla^l(n_2, u_1, u_2)\|_{L^2}^2 + \sum_{l=k}^{k+1} \|\nabla^l E\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 \right) \\ & \quad + \|(n_2, u_1, u_2)\|_{L^\infty} \|\nabla^{k+2}(n_2, u_1, u_2)\|_{L^2} \|\nabla^{k+2}(E, B)\|_{L^2}, \end{aligned} \quad (2.3)$$

where F is defined by

$$F = F(n_1, n_2, u_1, u_2, B) := \|\nabla n_1\|_{H^2} + \|(n_2, u_1, u_2)\|_{H^{k+1} \cap H^{\frac{k}{2}+2} \cap H^3} + \|\nabla B\|_{L^2}.$$

Proof. The standard ∇^l ($l = k, k+1, k+2$) energy estimates on the system (1.5) yield

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |\nabla^l(n_1, n_2, u_1, u_2)|^2 + \frac{d}{dt} \int |\nabla^l(E, B)|^2 + \nu \|\nabla^l(u_1, u_2)\|_{L^2}^2 \\ & = \int \nabla^l g_1 \nabla^l n_1 + \nabla^l g_2 \cdot \nabla^l u_1 + \nabla^l g_3 \nabla^l n_2 + \nabla^l g_4 \cdot \nabla^l u_2 \\ & \quad + \int \nabla^l(u_2 \times B) \cdot \nabla^l u_1 + \nabla^l(u_1 \times B) \cdot \nabla^l u_2 + 2\nu \int \nabla^l g_5 \cdot \nabla^l E \\ & := I_1 + I_2 + 2\nu I_3. \end{aligned} \quad (2.4)$$

We now estimate $I_1 \sim I_3$. First, by (1.6), we split I_1 as:

$$\begin{aligned} I_1 &= -\frac{1}{2} \int \nabla^l(u_1 \cdot \nabla n_1) \nabla^l n_1 + \nabla^l(u_1 \cdot \nabla u_1) \cdot \nabla^l u_1 \\ & \quad -\frac{1}{2} \int \nabla^l(u_1 \cdot \nabla n_2) \nabla^l n_2 + \nabla^l(u_1 \cdot \nabla u_2) \cdot \nabla^l u_2 \\ & \quad -\frac{1}{2} \int \nabla^l(u_2 \cdot \nabla n_2) \nabla^l n_1 + \nabla^l(u_2 \cdot \nabla n_1) \nabla^l n_2 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \int \nabla^l (u_2 \cdot \nabla u_2) \cdot \nabla^l u_1 + \nabla^l (u_2 \cdot \nabla u_1) \cdot \nabla^l u_2 \\
& -\frac{\mu}{2} \int \nabla^l (n_1 \operatorname{div} u_1) \nabla^l n_1 + \nabla^l (n_1 \nabla n_1) \cdot \nabla^l u_1 \\
& -\frac{\mu}{2} \int \nabla^l (n_1 \operatorname{div} u_2) \nabla^l n_2 + \nabla^l (n_1 \nabla n_2) \cdot \nabla^l u_2 \\
& -\frac{\mu}{2} \int \nabla^l (n_2 \operatorname{div} u_2) \nabla^l n_1 + \nabla^l (n_2 \nabla n_1) \cdot \nabla^l u_2 \\
& -\frac{\mu}{2} \int \nabla^l (n_2 \operatorname{div} u_1) \nabla^l n_2 + \nabla^l (n_2 \nabla n_2) \cdot \nabla^l u_1 \\
& := \frac{1}{2} (I_{11} + I_{12} + I_{13} + I_{14}) + \frac{\mu}{2} (I_{15} + I_{16} + I_{17} + I_{18}). \tag{2.5}
\end{aligned}$$

We shall estimate the eight terms on the right-hand side of (2.5). We must be careful about these terms involving n_1 since n_1 is degenerately dissipative. First we estimate I_{11} . We have to distinct the arguments by the value of l . For $l = k$ or $k + 1$, we have

$$\begin{aligned}
-\int \nabla^l (u_1 \cdot \nabla n_1) \nabla^l n_1 &= -\int \sum_{0 \leq \ell \leq l} C_l^\ell \nabla^{l-\ell} u_1 \cdot \nabla \nabla^\ell n_1 \nabla^l n_1 \\
&\lesssim \sum_{0 \leq \ell \leq l} \|\nabla^{l-\ell} u_1 \cdot \nabla \nabla^\ell n_1\|_{L^{6/5}} \|\nabla^l n_1\|_{L^6} \\
&\lesssim \sum_{0 \leq \ell \leq l} \|\nabla^{l-\ell} u_1 \cdot \nabla \nabla^\ell n_1\|_{L^{6/5}} \|\nabla^{l+1} n_1\|_{L^2}. \tag{2.6}
\end{aligned}$$

If $0 \leq \ell \leq [\frac{l}{2}]$, by Hölder's inequality and Lemma A.1, we have

$$\begin{aligned}
\|\nabla^{l-\ell} u_1 \cdot \nabla \nabla^\ell n_1\|_{L^{6/5}} &\lesssim \|\nabla^{l-\ell} u_1\|_{L^2} \|\nabla^{l+1} n_1\|_{L^3} \\
&\lesssim \|u_1\|_{L^2}^{\frac{\ell}{t}} \|\nabla^l u_1\|_{L^2}^{1-\frac{\ell}{t}} \|\nabla^\alpha n_1\|_{L^2}^{1-\frac{\ell}{t}} \|\nabla^{l+1} n_1\|_{L^2}^{\frac{\ell}{t}} \\
&\lesssim (\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}) (\|\nabla^{l+1} n_1\|_{L^2} + \|\nabla^l u_1\|_{L^2}), \tag{2.7}
\end{aligned}$$

where α is defined by

$$\ell + \frac{3}{2} = \alpha \times \left(1 - \frac{\ell}{l}\right) + (l+1) \times \frac{\ell}{l} \implies \alpha = \frac{3l-2\ell}{2l-2\ell} \in \left[\frac{3}{2}, 3\right) \text{ since } \ell \leq \frac{l}{2};$$

if $[\frac{l}{2}] + 1 \leq \ell \leq l$, by Hölder's inequality and Lemma A.1 again, we have

$$\begin{aligned}
\|\nabla^{l-\ell} u_1 \cdot \nabla \nabla^\ell n_1\|_{L^{6/5}} &\lesssim \|\nabla^{l-\ell} u_1\|_{L^3} \|\nabla^{l+1} n_1\|_{L^2} \\
&\lesssim \|\nabla^\alpha u_1\|_{L^2}^{\frac{\ell}{t}} \|\nabla^l u_1\|_{L^2}^{\frac{l-\ell}{t}} \|\nabla n_1\|_{L^2}^{\frac{l-\ell}{t}} \|\nabla^{l+1} n_1\|_{L^2}^{\frac{\ell}{t}} \\
&\lesssim (\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}) (\|\nabla^{l+1} n_1\|_{L^2} + \|\nabla^l u_1\|_{L^2}), \tag{2.8}
\end{aligned}$$

where α is defined by

$$l - \ell + \frac{1}{2} = \alpha \times \frac{\ell}{l} + l - \ell \implies \alpha = \frac{l}{2\ell} \in \left[\frac{1}{2}, 3\right) \text{ since } \ell \geq \frac{l+1}{2}.$$

In light of (2.7) and (2.8), we deduce from (2.6) that for $l = k$ or $k + 1$,

$$-\int \nabla^l (u_1 \cdot \nabla n_1) \nabla^l n_1 \lesssim (\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}) \left(\|\nabla^{l+1} n_1\|_{L^2}^2 + \|\nabla^l u_1\|_{L^2}^2 \right). \tag{2.9}$$

Now for $l = k + 2$, by integrating by parts and Lemma 2.3, we have

$$\begin{aligned}
-\int \nabla^{k+2} (u_1 \cdot \nabla n_1) \nabla^{k+2} n_1 &= -\int [\nabla^{k+2}, u_1] \cdot \nabla n_1 \nabla^{k+2} n_1 - \int u_1 \cdot \nabla \nabla^{k+2} n_1 \nabla^{k+2} n_1 \\
&\lesssim (\|\nabla u_1\|_{L^\infty} \|\nabla^{k+2} n_1\|_{L^2} + \|\nabla^{k+2} u_1\|_{L^2} \|\nabla n_1\|_{L^\infty}) \|\nabla^{k+2} n_1\|_{L^2} \\
&\quad -\frac{1}{2} \int u_1 \cdot \nabla (\nabla^{k+2} n_1 \nabla^{k+2} n_1) \\
&\lesssim \|\nabla(n_1, u_1)\|_{L^\infty} \|\nabla^{k+2}(n_1, u_1)\|_{L^2}^2 + \frac{1}{2} \int \operatorname{div} u_1 |\nabla^{k+2} n_1|^2 \\
&\lesssim (\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}) \|\nabla^{k+2}(n_1, u_1)\|_{L^2}^2. \tag{2.10}
\end{aligned}$$

On the other hand, like (2.10), we have for $l = k, k+1, k+2$,

$$\begin{aligned} - \int \nabla^l (u_1 \cdot \nabla u_1) \cdot \nabla^l u_1 &= - \int (u_1 \cdot \nabla \nabla^l u_1 + [\nabla^l, u_1] \cdot \nabla u_1) \cdot \nabla^l u_1 \\ &= - \int \frac{1}{2} u_1 \cdot \nabla (\nabla^l u_1 \cdot \nabla^l u_1) + [\nabla^l, u_1] \cdot \nabla u_1 \cdot \nabla^l u_1 \\ &\lesssim \|\nabla u_1\|_{L^\infty} \|\nabla^l u_1\|_{L^2}^2. \end{aligned} \quad (2.11)$$

Hence, by (2.9)–(2.11), we have for $l = k, k+1$,

$$I_{11} \lesssim (\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}) \left(\|\nabla^{l+1} n_1\|_{L^2}^2 + \|\nabla^l u_1\|_{L^2}^2 \right),$$

and for $l = k+2$,

$$I_{11} \lesssim (\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}) \|\nabla^{k+2}(n_1, u_1)\|_{L^2}^2.$$

Like (2.10), we have for $l = k, k+1, k+2$,

$$I_{12} \lesssim \|(n_2, u_1, u_2)\|_{H^3} \|\nabla^l(n_2, u_1, u_2)\|_{L^2}^2, \quad I_{14} \lesssim \|(u_1, u_2)\|_{H^3} \|\nabla^l(u_1, u_2)\|_{L^2}^2.$$

As in (2.6)–(2.10), we have for $l = k, k+1$,

$$I_{13} \lesssim (\|\nabla n_1\|_{H^2} + \|(n_2, u_2)\|_{H^3}) \left(\|\nabla^{l+1}(n_1, n_2)\|_{L^2}^2 + \|\nabla^l u_1\|_{L^2}^2 \right),$$

and for $l = k+2$,

$$I_{13} \lesssim (\|\nabla n_1\|_{H^2} + \|(n_2, u_2)\|_{H^3}) \|\nabla^{k+2}(n_1, n_2, u_1)\|_{L^2}^2.$$

We next estimate the term I_{15} . For $l = k$ or $k+1$, we split I_{15} as:

$$\begin{aligned} I_{15} &= - \int \nabla^l (n_1 \operatorname{div} u_1) \nabla^l n_1 + \nabla^l (n_1 \nabla n_1) \cdot \nabla^l u_1 \\ &= - \int \sum_{0 \leq \ell \leq l} C_l^\ell \left(\nabla^{l-\ell} n_1 \nabla^\ell \operatorname{div} u_1 \nabla^l n_1 + \nabla^{l-\ell} n_1 \nabla^{\ell+1} n_1 \cdot \nabla^l u_1 \right) \\ &= - \int \sum_{0 \leq \ell \leq l-1} C_l^\ell \nabla^{l-\ell} n_1 \nabla^\ell \operatorname{div} u_1 \nabla^l n_1 - \int \sum_{0 \leq \ell \leq l-1} C_l^\ell \nabla^{l-\ell} n_1 \nabla^{\ell+1} n_1 \cdot \nabla^l u_1 \\ &\quad - \int n_1 \operatorname{div} \nabla^l u_1 \nabla^l n_1 + n_1 \nabla^{l+1} n_1 \cdot \nabla^l u_1 \\ &:= I_{151} + I_{152} + I_{153}. \end{aligned} \quad (2.12)$$

First we estimate I_{153} . By Hölder's, Sobolev's and Cauchy's inequalities, we obtain

$$\begin{aligned} I_{153} &= - \int n_1 \operatorname{div} \nabla^l u_1 \nabla^l n_1 + n_1 \nabla^{l+1} n_1 \cdot \nabla^l u_1 = - \int n_1 \operatorname{div} (\nabla^l u_1 \nabla^l n_1) = \int \nabla n_1 \nabla^l u_1 \nabla^l n_1 \\ &\lesssim \|\nabla n_1\|_{L^3} \|\nabla^l u_1\|_{L^2} \|\nabla^l n_1\|_{L^6} \lesssim \|\nabla n_1\|_{H^2} \left(\|\nabla^{l+1} n_1\|_{L^2}^2 + \|\nabla^l u_1\|_{L^2}^2 \right). \end{aligned} \quad (2.13)$$

Next we estimate the term I_{151} . By Hölder's and Sobolev's inequalities, we obtain

$$\begin{aligned} I_{151} &= - \int \sum_{0 \leq \ell \leq l-1} C_l^\ell \nabla^{l-\ell} n_1 \nabla^\ell \operatorname{div} u_1 \nabla^l n_1 \lesssim \sum_{0 \leq \ell \leq l-1} \|\nabla^{l-\ell} n_1 \nabla^\ell \operatorname{div} u_1\|_{L^{6/5}} \|\nabla^l n_1\|_{L^6} \\ &\lesssim \sum_{0 \leq \ell \leq l-1} \|\nabla^{l-\ell} n_1 \nabla^\ell \operatorname{div} u_1\|_{L^{6/5}} \|\nabla^{l+1} n_1\|_{L^2}. \end{aligned} \quad (2.14)$$

If $0 \leq \ell \leq [\frac{l}{2}]$, by Hölder's inequality and Lemma A.1, we have

$$\begin{aligned} \|\nabla^{l-\ell} n_1 \nabla^\ell \operatorname{div} u_1\|_{L^{6/5}} &\lesssim \|\nabla^{l-\ell} n_1\|_{L^3} \|\nabla^{\ell+1} u_1\|_{L^2} \\ &\lesssim \|\nabla n_1\|_{L^2}^{\frac{2\ell+1}{2l}} \|\nabla^{l+1} n_1\|_{L^2}^{\frac{2l-2\ell-1}{2l}} \|\nabla^\alpha u_1\|_{L^2}^{\frac{2l-2\ell-1}{2l}} \|\nabla^l u_1\|_{L^2}^{\frac{2\ell+1}{2l}} \\ &\lesssim (\|\nabla n_1\|_{L^2} + \|u_1\|_{H^3}) \left(\|\nabla^{l+1} n_1\|_{L^2} + \|\nabla^l u_1\|_{L^2} \right), \end{aligned} \quad (2.15)$$

where α is defined by

$$\ell + 1 = \alpha \times \frac{2l - 2\ell - 1}{2l} + \frac{2\ell + 1}{2} \implies \alpha = \frac{l}{2l - 2\ell - 1} \in \left(\frac{1}{2}, 3 \right) \text{ since } \ell \leq \frac{l}{2};$$

if $\lceil \frac{l}{2} \rceil + 1 \leq \ell \leq l-1$, by Hölder's inequality and Lemma A.1 again, we have

$$\begin{aligned} \|\nabla^{l-\ell} n_1 \nabla^\ell \operatorname{div} u_1\|_{L^{6/5}} &\lesssim \|\nabla^{l-\ell} n_1\|_{L^3} \|\nabla^{\ell+1} u_1\|_{L^2} \\ &\lesssim \|\nabla^\alpha n_1\|_{L^2}^{\frac{\ell+1}{l}} \|\nabla^{l+1} n_1\|_{L^2}^{\frac{l-\ell-1}{l}} \|u_1\|_{L^2}^{\frac{l-\ell-1}{l}} \|\nabla^l u_1\|_{L^2}^{\frac{\ell+1}{l}} \\ &\lesssim (\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}) (\|\nabla^{l+1} n_1\|_{L^2} + \|\nabla^l u_1\|_{L^2}), \end{aligned} \quad (2.16)$$

where α is defined by

$$\begin{aligned} l - \ell + \frac{1}{2} &= \alpha \times \frac{\ell+1}{l} + (l+1) \times \frac{l-\ell-1}{l} \\ \implies \alpha &= 1 + \frac{l}{2\ell+2} \in \left[\frac{3}{2}, 3\right) \text{ since } \ell \geq \frac{l+1}{2}. \end{aligned}$$

In light of (2.15) and (2.16), we deduce from (2.14) that

$$I_{151} \lesssim (\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}) \left(\|\nabla^{l+1} n_1\|_{L^2}^2 + \|\nabla^l u_1\|_{L^2}^2 \right). \quad (2.17)$$

Finally, we estimate the term I_{152} . By Hölder's inequality, we obtain

$$I_{152} = - \int \sum_{0 \leq \ell \leq l-1} C_l^\ell \nabla^{l-\ell} n_1 \nabla^{\ell+1} n_1 \cdot \nabla^l u_1 \lesssim \sum_{0 \leq \ell \leq l-1} \|\nabla^{l-\ell} n_1 \nabla^{\ell+1} n_1\|_{L^2} \|\nabla^l u_1\|_{L^2}. \quad (2.18)$$

If $0 \leq \ell \leq \lceil \frac{l}{2} \rceil$, by Hölder's inequality and Lemma A.1, we have

$$\begin{aligned} \|\nabla^{l-\ell} n_1 \nabla^{\ell+1} n_1\|_{L^2} &\lesssim \|\nabla^{l-\ell} n_1\|_{L^6} \|\nabla^{\ell+1} n_1\|_{L^3} \\ &\lesssim \|\nabla n_1\|_{L^2}^{\frac{\ell}{l}} \|\nabla^{l+1} n_1\|_{L^2}^{\frac{l-\ell}{l}} \|\nabla^\alpha n_1\|_{L^2}^{\frac{l-\ell}{l}} \|\nabla^{l+1} n_1\|_{L^2}^{\frac{\ell}{l}} \\ &\lesssim \|\nabla n_1\|_{H^2} \|\nabla^{l+1} n_1\|_{L^2}, \end{aligned} \quad (2.19)$$

where α is defined by

$$\ell + \frac{3}{2} = \alpha \times \frac{l-\ell}{l} + (l+1) \times \frac{\ell}{l} \implies \alpha = \frac{3l-2\ell}{2l-2\ell} \in \left[\frac{3}{2}, 3\right) \text{ since } \ell \leq \frac{l}{2};$$

if $\lceil \frac{l}{2} \rceil + 1 \leq \ell \leq l-1$, by Hölder's inequality and Lemma A.1 again, we have

$$\begin{aligned} \|\nabla^{l-\ell} n_1 \nabla^{\ell+1} n_1\|_{L^2} &\lesssim \|\nabla^{l-\ell} n_1\|_{L^3} \|\nabla^{\ell+1} n_1\|_{L^6} \\ &\lesssim \|\nabla^\alpha n_1\|_{L^2}^{\frac{\ell}{l-1}} \|\nabla^{l+1} n_1\|_{L^2}^{1-\frac{\ell}{l-1}} \|\nabla^2 n_1\|_{L^2}^{1-\frac{\ell}{l-1}} \|\nabla^{l+1} n_1\|_{L^2}^{\frac{\ell}{l-1}} \\ &\lesssim \|\nabla n_1\|_{H^2} \|\nabla^{l+1} n_1\|_{L^2}, \end{aligned} \quad (2.20)$$

where α is defined by

$$\begin{aligned} l - \ell + \frac{1}{2} &= \alpha \times \frac{\ell}{l-1} + (l+1) \times \left(1 - \frac{\ell}{l-1}\right) \\ \implies \alpha &= 2 + \frac{-l+1}{2\ell} \in \left[\frac{3}{2}, 3\right) \text{ since } \ell \geq \frac{l+1}{2}. \end{aligned}$$

In light of (2.19) and (2.20), we deduce from (2.18) that

$$I_{152} \lesssim \|\nabla n_1\|_{H^2} \left(\|\nabla^{l+1} n_1\|_{L^2}^2 + \|\nabla^l u_1\|_{L^2}^2 \right). \quad (2.21)$$

Hence, by (2.13), (2.17) and (2.21), we deduce from (2.12) that for $l = k, k+1$,

$$I_{15} \lesssim (\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}) \left(\|\nabla^{l+1} n_1\|_{L^2}^2 + \|\nabla^l u_1\|_{L^2}^2 \right).$$

For $l = k + 2$, like (2.10), we have

$$\begin{aligned}
I_{15} &= - \int \nabla^{k+2}(n_1 \operatorname{div} u_1) \nabla^{k+2} n_1 + \nabla^{k+2}(n_1 \nabla n_1) \cdot \nabla^{k+2} u_1 \\
&= - \int [\nabla^{k+2}, n_1] \operatorname{div} u_1 \nabla^{k+2} n_1 + [\nabla^{k+2}, n_1] \nabla n_1 \cdot \nabla^{k+2} u_1 \\
&\quad - \int n_1 \operatorname{div} \nabla^{k+2} u_1 \nabla^{k+2} n_1 + n_1 \nabla \nabla^{k+2} n_1 \cdot \nabla^{k+2} u_1 \\
&\lesssim (\|\nabla n_1\|_{L^\infty} \|\nabla^{k+2} u_1\|_{L^2} + \|\nabla^{k+2} n_1\|_{L^2} \|\nabla u_1\|_{L^\infty}) \|\nabla^{k+2} n_1\|_{L^2} \\
&\quad - \int n_1 \operatorname{div} (\nabla^{k+2} u_1 \nabla^{k+2} n_1) \\
&\lesssim \|\nabla(n_1, u_1)\|_{L^\infty} \|\nabla^{k+2}(n_1, u_1)\|_{L^2}^2 + \int \nabla n_1 \nabla^{k+2} u_1 \nabla^{k+2} n_1 \\
&\lesssim (\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}) \|\nabla^{k+2}(n_1, u_1)\|_{L^2}^2.
\end{aligned}$$

Applying the same arguments to these terms $I_{16}-I_{18}$, we deduce that for $l = k$ or $k + 1$,

$$I_{16} + I_{17} + I_{18} \lesssim (\|\nabla n_1\|_{H^2} + \|(n_2, u_1, u_2)\|_{H^3}) \left(\|\nabla^{l+1}(n_1, n_2)\|_{L^2}^2 + \|\nabla^l(u_1, u_2)\|_{L^2}^2 \right);$$

for $l = k + 2$,

$$I_{16} + I_{17} + I_{18} \lesssim (\|\nabla n_1\|_{H^2} + \|(n_2, u_1, u_2)\|_{H^3}) \|\nabla^{k+2}(n_1, n_2, u_1, u_2)\|_{L^2}^2.$$

Hence, by these estimates for $I_{11} \sim I_{18}$, we deduce for $l = k, k + 1$

$$I_1 \lesssim (\|\nabla n_1\|_{H^2} + \|(n_2, u_1, u_2)\|_{H^3}) \left(\|\nabla^{l+1}(n_1, n_2)\|_{L^2}^2 + \|\nabla^l(u_1, u_2)\|_{L^2}^2 \right); \quad (2.22)$$

for $l = k + 2$

$$I_1 \lesssim (\|\nabla n_1\|_{H^2} + \|(n_2, u_1, u_2)\|_{H^3}) \|\nabla^{k+2}(n_1, n_2, u_1, u_2)\|_{L^2}^2.$$

Now we estimate the term I_2 , and we must be much more careful about this term since the magnetic field B has the weakest dissipative estimates. First of all, we have

$$\begin{aligned}
I_2 &= \sum_{\ell=1}^l C_\ell \int \nabla^{l-\ell} u_2 \times \nabla^\ell B \cdot \nabla^l u_1 + \nabla^{l-\ell} u_1 \times \nabla^\ell B \cdot \nabla^l u_2 \\
&\lesssim C_l \sum_{\ell=1}^l (\|\nabla^{l-\ell} u_2 \nabla^\ell B\|_{L^2} \|\nabla^l u_1\|_{L^2} + \|\nabla^{l-\ell} u_1 \nabla^\ell B\|_{L^2} \|\nabla^l u_2\|_{L^2}). \quad (2.23)
\end{aligned}$$

Here we notice $\nabla^l u_2 \times B \cdot \nabla^l u_1 + \nabla^l u_1 \times B \cdot \nabla^l u_2 = 0$. We only estimate the first term on the right-hand side of (2.23), the second term can be estimated similarly. We again have to distinct the arguments by the value of l . First, let $l = k$. We take $L^3 - L^6$ and then apply Lemma 2.1 to have

$$\begin{aligned}
\|\nabla^{k-\ell} u_2 \nabla^\ell B\|_{L^2} &\lesssim \|\nabla^{k-\ell} u_2\|_{L^3} \|\nabla^\ell B\|_{L^6} \\
&\lesssim \|\nabla^\alpha u_2\|_{L^2}^{\frac{\ell}{k}} \|\nabla^k u_2\|_{L^2}^{1-\frac{\ell}{k}} \|\nabla B\|_{L^2}^{1-\frac{\ell}{k}} \|\nabla^{k+1} B\|_{L^2}^{\frac{\ell}{k}},
\end{aligned}$$

where α is defined by

$$k - \ell + \frac{1}{2} = \alpha \times \frac{\ell}{k} + k \times \left(1 - \frac{\ell}{k}\right) \implies \alpha = \frac{k}{2\ell} \leq \frac{k}{2}.$$

Hence by Young's inequality, we have that for $l = k$,

$$I_2 \leq C_k \left(\|(u_1, u_2)\|_{H^{\frac{k}{2}}} + \|\nabla B\|_{L^2} \right) \left(\|\nabla^k(u_1, u_2)\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 \right). \quad (2.24)$$

We then let $l = k + 1$. If $1 \leq \ell \leq k$, we take $L^3 - L^6$ and by Lemma 2.1 again to obtain

$$\begin{aligned}
\|\nabla^{k+1-\ell} u_2 \nabla^\ell B\|_{L^2} &\lesssim \|\nabla^{k+1-\ell} u_2\|_{L^3} \|\nabla^\ell B\|_{L^6} \\
&\lesssim \|\nabla^\alpha u_2\|_{L^2}^{\frac{\ell}{k}} \|\nabla^{k+1} u_2\|_{L^2}^{1-\frac{\ell}{k}} \|\nabla B\|_{L^2}^{1-\frac{\ell}{k}} \|\nabla^{k+1} B\|_{L^2}^{\frac{\ell}{k}},
\end{aligned}$$

where α is defined by

$$k + 1 - \ell + \frac{1}{2} = \alpha \times \frac{\ell}{k} + (k + 1) \times \left(1 - \frac{\ell}{k}\right) \implies \alpha = 1 + \frac{k}{2\ell} \leq \frac{k}{2} + 1;$$

if $\ell = k + 1$, we take $L^\infty - L^2$ to get

$$\|u_2 \nabla^{k+1} B\|_{L^2} \lesssim \|u_2\|_{L^\infty} \|\nabla^{k+1} B\|_{L^2}.$$

We thus have that for $l = k + 1$, by Sobolev's inequality,

$$I_2 \leq C_k \left(\|(u_1, u_2)\|_{H^{\frac{k}{2}+1} \cap H^2} + \|\nabla B\|_{L^2} \right) \left(\|\nabla^{k+1}(u_1, u_2)\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 \right).$$

We now let $l = k + 2$. If $1 \leq \ell \leq k$, we take $L^3 - L^6$ and by Lemma 2.1 again to have

$$\begin{aligned} \|\nabla^{k+2-\ell} u_2 \nabla^\ell B\|_{L^2} &\lesssim \|\nabla^{k+2-\ell} u_2\|_{L^3} \|\nabla^\ell B\|_{L^6} \\ &\lesssim \|\nabla^\alpha u_2\|_{L^2}^{\frac{\ell}{k}} \|\nabla^{k+2} u_2\|_{L^2}^{1-\frac{\ell}{k}} \|\nabla B\|_{L^2}^{1-\frac{\ell}{k}} \|\nabla^{k+1} B\|_{L^2}^{\frac{\ell}{k}}, \end{aligned}$$

where α is defined by

$$k + 2 - \ell + \frac{1}{2} = \alpha \times \frac{\ell}{k} + (k + 2) \times \left(1 - \frac{\ell}{k}\right) \implies \alpha = 2 + \frac{k}{2\ell} \leq \frac{k}{2} + 2;$$

if $\ell = k + 1$ or $k + 2$, we take $L^\infty - L^2$ to get

$$\|\nabla u_2 \nabla^{k+1} B\|_{L^2} \lesssim \|\nabla u_2\|_{L^\infty} \|\nabla^{k+1} B\|_{L^2},$$

and

$$\|u_2 \nabla^{k+2} B\|_{L^2} \lesssim \|u_2\|_{L^\infty} \|\nabla^{k+2} B\|_{L^2}.$$

We thus have that for $l = k + 2$,

$$\begin{aligned} I_2 &\leq C_k \left(\|(u_1, u_2)\|_{H^{\frac{k}{2}+2} \cap H^3} + \|\nabla B\|_{L^2} \right) \left(\|\nabla^{k+2}(u_1, u_2)\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 \right) \\ &\quad + C \|(u_1, u_2)\|_{L^\infty} \|\nabla^{k+2} B\|_{L^2} \|\nabla^{k+2}(u_1, u_2)\|_{L^2}. \end{aligned}$$

We now estimate the last term I_3 in (2.4). First, we split I_3 as:

$$\begin{aligned} I_3 &= \nu \sum_{\ell=0}^l C_l^\ell \int \left[\nabla^\ell f \left(\frac{n_1 - n_2}{2} \right) \nabla^{l-\ell} \left(\frac{u_1 - u_2}{2} \right) - \nabla^\ell f \left(\frac{n_1 + n_2}{2} \right) \nabla^{l-\ell} \left(\frac{u_1 + u_2}{2} \right) \right] \cdot \nabla^l E \\ &= \frac{\nu}{2} \sum_{\ell=0}^l C_l^\ell \int \nabla^\ell f \left(\frac{n_1 - n_2}{2} \right) \nabla^{l-\ell} u_1 \cdot \nabla^l E - \frac{\nu}{2} \sum_{\ell=0}^l C_l^\ell \int \nabla^\ell f \left(\frac{n_1 - n_2}{2} \right) \nabla^{l-\ell} u_2 \cdot \nabla^l E \\ &\quad - \frac{\nu}{2} \sum_{\ell=0}^l C_l^\ell \int \nabla^\ell f \left(\frac{n_1 + n_2}{2} \right) \nabla^{l-\ell} u_1 \cdot \nabla^l E + \frac{\nu}{2} \sum_{\ell=0}^l C_l^\ell \int \nabla^\ell f \left(\frac{n_1 + n_2}{2} \right) \nabla^{l-\ell} u_2 \cdot \nabla^l E \\ &:= \frac{\nu}{2} I_{31} + \frac{\nu}{2} I_{32} + \frac{\nu}{2} I_{33} + \frac{\nu}{2} I_{34}. \end{aligned} \tag{2.25}$$

We still have to distinct the arguments by the value of l . For $l = k$ or $k + 1$, we only estimate the first term I_{31} on the right-hand side of (2.25), the other terms $I_{32}-I_{34}$ can be estimated similarly. If $0 \leq \ell \leq l - 1$, we take $L^\infty - L^2$ and by Lemma 2.1 and the estimate (2.2) of Lemma 2.2 to obtain

$$\begin{aligned} &\left\| \nabla^\ell f \left(\frac{n_1 - n_2}{2} \right) \nabla^{l-\ell} u_1 \right\|_{L^2} \\ &\leq \left\| \nabla^\ell f \left(\frac{n_1 - n_2}{2} \right) \right\|_{L^\infty} \|\nabla^{l-\ell} u_1\|_{L^2} \\ &\leq C_l \|\nabla^{\ell+1} n_1\|_{L^2}^{\frac{1}{2}} \|\nabla^{\ell+2} n_1\|_{L^2}^{\frac{1}{2}} \|\nabla^{l-\ell} u_1\|_{L^2} + C_l \|\nabla^{\ell+1} n_2\|_{L^2}^{\frac{1}{2}} \|\nabla^{\ell+2} n_2\|_{L^2}^{\frac{1}{2}} \|\nabla^{l-\ell} u_1\|_{L^2} \\ &\leq C_l \left(\|\nabla n_1\|_{L^2}^{\frac{l-\ell}{l}} \|\nabla^{l+1} n_1\|_{L^2}^{\frac{\ell}{l}} \right)^{\frac{1}{2}} \left(\|\nabla n_1\|_{L^2}^{\frac{l-\ell-1}{l}} \|\nabla^{l+1} n_1\|_{L^2}^{\frac{\ell+1}{l}} \right)^{\frac{1}{2}} \|\nabla^{l-\ell} u_1\|_{L^2} \\ &\quad + C_l \left(\|\nabla n_2\|_{L^2}^{\frac{l-\ell}{l}} \|\nabla^{l+1} n_2\|_{L^2}^{\frac{\ell}{l}} \right)^{\frac{1}{2}} \left(\|\nabla n_2\|_{L^2}^{\frac{l-\ell-1}{l}} \|\nabla^{l+1} n_2\|_{L^2}^{\frac{\ell+1}{l}} \right)^{\frac{1}{2}} \|\nabla^{l-\ell} u_1\|_{L^2} \\ &\leq C_l \|\nabla n_1\|_{L^2}^{1-\frac{2\ell+1}{2l}} \|\nabla^{l+1} n_1\|_{L^2}^{\frac{2\ell+1}{2l}} \|\nabla^\alpha u_1\|_{L^2}^{\frac{2\ell+1}{2l}} \|\nabla^l u_1\|_{L^2}^{1-\frac{2\ell+1}{2l}} \\ &\quad + C_l \|\nabla n_2\|_{L^2}^{1-\frac{2\ell+1}{2l}} \|\nabla^{l+1} n_2\|_{L^2}^{\frac{2\ell+1}{2l}} \|\nabla^\alpha u_1\|_{L^2}^{\frac{2\ell+1}{2l}} \|\nabla^l u_1\|_{L^2}^{1-\frac{2\ell+1}{2l}}, \end{aligned}$$

where α is defined by

$$l - \ell = \alpha \times \frac{2\ell + 1}{2l} + l \times \left(1 - \frac{2\ell + 1}{2l}\right) \implies \alpha = \frac{l}{2\ell + 1} \leq l;$$

if $\ell = l$, we take $L^2 - L^\infty$ and by the estimate (2.2) of Lemma 2.2 to have

$$\left\| \nabla^l f \left(\frac{n_1 - n_2}{2} \right) u_1 \right\|_{L^2} \leq \left\| \nabla^l f \left(\frac{n_1 - n_2}{2} \right) \right\|_{L^6} \|u_1\|_{L^3} \leq C_l \|\nabla^{l+1}(n_1, n_2)\|_{L^2} \|u_1\|_{H^1}.$$

We thus have that for $l = k$ or $k + 1$,

$$I_{31} \leq C_l (\|\nabla(n_1, n_2)\|_{L^2} + \|u_1\|_{H^l \cap H^1}) \left(\|\nabla^{l+1}(n_1, n_2)\|_{L^2}^2 + \|\nabla^l u_1\|_{L^2}^2 + \|\nabla^l E\|_{L^2}^2 \right).$$

Hence, we have that for $l = k$ or $k + 1$,

$$I_3 \leq C_l (\|\nabla(n_1, n_2)\|_{L^2} + \|(u_1, u_2)\|_{H^l \cap H^1}) \left(\|\nabla^{l+1}(n_1, n_2)\|_{L^2}^2 + \|\nabla^l(u_1, u_2)\|_{L^2}^2 + \|\nabla^l E\|_{L^2}^2 \right). \quad (2.26)$$

Now for $l = k + 2$, we rewrite $I_{31} + I_{33}$ as

$$\begin{aligned} I_{31} + I_{33} &= \sum_{\ell=0}^{k+2} C_{k+2}^\ell \int \nabla^\ell g \nabla^{k+2-\ell} u_1 \cdot \nabla^{k+2} E \\ &= \int (g \nabla^{k+2} u_1 + \nabla^{k+2} g u_1) \cdot \nabla^{k+2} E - \sum_{\ell=1}^{k+1} C_{k+2}^\ell \int \nabla (\nabla^{k+2-\ell} g \nabla^\ell u_1) \cdot \nabla^{k+1} E \\ &= \int (g \nabla^{k+2} u_1 + \nabla^{k+2} g u_1) \cdot \nabla^{k+2} E - (k+2) \int (\nabla^{k+2} g \nabla u_1 + \nabla g \nabla^{k+2} u_1) \cdot \nabla^{k+1} E \\ &\quad - \sum_{\ell=2}^{k+1} C_{k+2}^\ell \int \nabla^{k+3-\ell} g \nabla^\ell u_1 \cdot \nabla^{k+1} E - \sum_{\ell=1}^k C_{k+2}^\ell \int \nabla^{k+2-\ell} g \nabla^{\ell+1} u_1 \cdot \nabla^{k+1} E \\ &:= I_{311} + I_{312} + I_{313} + I_{314}, \end{aligned}$$

where the function g is defined as

$$g := f \left(\frac{n_1 - n_2}{2} \right) - f \left(\frac{n_1 + n_2}{2} \right). \quad (2.27)$$

By Lemma 2.2 and (2.1), we have

$$\begin{aligned} I_{311} &\leq C_k (\|g\|_{L^\infty} \|\nabla^{k+2} u_1\|_{L^2} + \|\nabla^{k+2} g\|_{L^2} \|u_1\|_{L^\infty}) \|\nabla^{k+2} E\|_{L^2} \\ &\leq C_k \|(n_2, u_1)\|_{L^\infty} \|\nabla^{k+2}(n_2, u_1)\|_{L^2} \|\nabla^{k+2} E\|_{L^2} \end{aligned}$$

and

$$\begin{aligned} I_{312} &\leq C_k (\|\nabla^{k+2} g\|_{L^2} \|\nabla u_1\|_{L^\infty} + \|\nabla g\|_{L^\infty} \|\nabla^{k+2} u_1\|_{L^2}) \|\nabla^{k+1} E\|_{L^2} \\ &\leq C_k \|\nabla(n_2, u_1)\|_{L^\infty} \|\nabla^{k+2}(n_2, u_1)\|_{L^2} \|\nabla^{k+1} E\|_{L^2}. \end{aligned}$$

As for the cases $l = k, k + 1$ for I_3 , we can bound I_{313} and I_{314} by

$$\begin{aligned} I_{313} + I_{314} &\leq C_k (\|\nabla(n_1, n_2)\|_{L^2} + \|u_1\|_{H^{k+1}}) \left(\|\nabla^{k+2}(n_1, n_2)\|_{L^2}^2 + \|\nabla^{k+1} u_1\|_{L^2}^2 + \|\nabla^{k+1} E\|_{L^2}^2 \right). \end{aligned}$$

Hence, we have that for $l = k + 2$,

$$\begin{aligned} I_{31} + I_{33} &\leq C_k (\|\nabla n_1\|_{L^2} + \|(n_2, u_1)\|_{H^{k+1} \cap H^3}) \left(\|\nabla^{k+1} u_1\|_{L^2}^2 + \|\nabla^{k+2}(n_1, n_2, u_1)\|_{L^2}^2 + \|\nabla^{k+1} E\|_{L^2}^2 \right) \\ &\quad + C_k \|(n_2, u_1)\|_{L^\infty} \|\nabla^{k+2}(n_2, u_1)\|_{L^2} \|\nabla^{k+2} E\|_{L^2}. \end{aligned}$$

Similarly, we can estimate $I_{32} + I_{34}$ for $l = k + 2$. So, we have for $l = k + 2$,

$$\begin{aligned} I_3 &\leq C_k (\|\nabla n_1\|_{L^2} + \|(n_2, u_1, u_2)\|_{H^{k+1} \cap H^3}) \left(\sum_{l=k+1}^{k+2} \|\nabla^l(n_1, n_2, u_1, u_2)\|_{L^2}^2 + \|\nabla^{k+1} E\|_{L^2}^2 \right) \\ &\quad + C_k \|(n_2, u_1, u_2)\|_{L^\infty} \|\nabla^{k+2}(n_2, u_1, u_2)\|_{L^2} \|\nabla^{k+2} E\|_{L^2}. \end{aligned}$$

Consequently, plugging these estimates for $I_1 \sim I_3$ into (2.4) with $l = k, k + 1, k + 2$, and then summing up, we deduce (2.3). \square

Note that in Lemma 2.8 we only derive the dissipative estimates of u_1 and u_2 . We now recover the dissipative estimates of n_1, n_2, E and B by constructing some interactive energy functionals in the following lemma.

Lemma 2.9. *For any integer $k \geq 0$, we have that for any small fixed $\eta > 0$,*

$$\begin{aligned}
& \frac{d}{dt} \left(\sum_{l=k}^{k+1} \int \nabla^l u_1 \cdot \nabla \nabla^l n_1 + \nabla^l u_2 \cdot \nabla \nabla^l n_2 - \sum_{l=k}^{k+1} \int \nabla^l u_2 \cdot \nabla^l E - \eta \int \nabla^k E \cdot \nabla^k \nabla \times B \right) \\
& + \lambda \left(\sum_{l=k+1}^{k+2} \|\nabla^l n_1\|_{L^2}^2 + \sum_{l=k}^{k+2} \|\nabla^l n_2\|_{L^2}^2 + \sum_{l=k}^{k+1} \|\nabla^l E\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 \right) \\
& \leq C \sum_{l=k}^{k+2} \|\nabla^l(u_1, u_2)\|_{L^2}^2 \\
& + C_k G \left(\sum_{l=k+1}^{k+2} \|\nabla^l n_1\|_{L^2}^2 + \sum_{l=k}^{k+2} \|\nabla^l(n_2, u_1, u_2)\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 \right), \tag{2.28}
\end{aligned}$$

where G is defined by

$$G = G(n_1, n_2, u_1, u_2, B) := \|\nabla n_1\|_{H^2}^2 + \|(n_2, u_1, u_2)\|_{H^{k+1} \cap H^3}^2 + \|\nabla B\|_{L^2}^2.$$

Proof. We divide the proof into four steps.

Step 1: Dissipative estimates of n_1, n_2 .

Applying ∇^l ($l = k, k+1$) to (1.5)₂, (1.5)₄ and then taking the L^2 inner product with $\nabla \nabla^l n_1, \nabla \nabla^l n_2$ respectively, we obtain

$$\begin{aligned}
& \int \partial_t \nabla^l u_1 \cdot \nabla \nabla^l n_1 + \partial_t \nabla^l u_2 \cdot \nabla \nabla^l n_2 + \|\nabla \nabla^l(n_1, n_2)\|_{L^2}^2 \\
& \leq 2\nu \int \nabla^l E \cdot \nabla \nabla^l n_2 + C \|\nabla^l u_1\|_{L^2} \|\nabla^{l+1} n_1\|_{L^2} + C \|\nabla^l u_2\|_{L^2} \|\nabla^{l+1} n_2\|_{L^2} \\
& + \|\nabla^l(u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 + n_1 \nabla n_2 + n_2 \nabla n_1 + u_1 \times B)\|_{L^2} \|\nabla^{l+1} n_2\|_{L^2} \\
& + \|\nabla^l(u_1 \cdot \nabla u_1 + u_2 \cdot \nabla u_2 + n_1 \nabla n_1 + n_2 \nabla n_2 + u_2 \times B)\|_{L^2} \|\nabla^{l+1} n_1\|_{L^2}. \tag{2.29}
\end{aligned}$$

The delicate first term on the left-hand side of (2.29) involves $\partial_t \nabla^l(u_1, u_2)$, and the key idea is to integrate by parts in the t -variable and use the equations (1.5)₁ and (1.5)₃. Thus integrating by parts for both the t - and x -variables, we obtain

$$\begin{aligned}
& \int \nabla^l \partial_t u_1 \cdot \nabla \nabla^l n_1 + \nabla^l \partial_t u_2 \cdot \nabla \nabla^l n_2 \\
& = \frac{d}{dt} \int \nabla^l u_1 \cdot \nabla \nabla^l n_1 + \nabla^l u_2 \cdot \nabla \nabla^l n_2 - \int \nabla^l u_1 \cdot \nabla \nabla^l \partial_t n_1 + \nabla^l u_2 \cdot \nabla \nabla^l \partial_t n_2 \\
& = \frac{d}{dt} \int \nabla^l u_1 \cdot \nabla \nabla^l n_1 + \nabla^l u_2 \cdot \nabla \nabla^l n_2 + \int \nabla^l \operatorname{div} u_1 \nabla^l \partial_t n_1 + \nabla^l \operatorname{div} u_2 \nabla^l \partial_t n_2 \\
& = \frac{d}{dt} \int \nabla^l u_1 \cdot \nabla \nabla^l n_1 + \nabla^l u_2 \cdot \nabla \nabla^l n_2 - \|\nabla^l \operatorname{div}(u_1, u_2)\|_{L^2}^2 + \int \nabla^l \operatorname{div} u_1 \nabla^l g_1 + \nabla^l \operatorname{div} u_2 \nabla^l g_3 \\
& \geq \frac{d}{dt} \int \nabla^l u_1 \cdot \nabla \nabla^l n_1 + \nabla^l u_2 \cdot \nabla \nabla^l n_2 - C \|\nabla^{l+1}(u_1, u_2)\|_{L^2}^2 \\
& - C \|\nabla^l(u_1 \cdot \nabla n_1, u_2 \cdot \nabla n_1, n_1 \operatorname{div} u_1, n_1 \operatorname{div} u_2)\|_{L^2}^2 \\
& - C \|\nabla^l(u_1 \cdot \nabla n_2, u_2 \cdot \nabla n_2, n_2 \operatorname{div} u_2, n_2 \operatorname{div} u_1)\|_{L^2}^2.
\end{aligned}$$

First, we have

$$\|\nabla^l(u_1 \cdot \nabla n_1)\|_{L^2} \lesssim \sum_{0 \leq \ell \leq l} \|\nabla^\ell u_1 \cdot \nabla \nabla^{l-\ell} n_1\|_{L^2}.$$

If $\ell = 0$, then

$$\|u_1 \cdot \nabla \nabla^l n_1\|_{L^2} \lesssim \|u_1\|_{L^\infty} \|\nabla^{l+1} n_1\|_{L^2} \lesssim \|u_1\|_{H^2} \|\nabla^{l+1} n_1\|_{L^2}; \tag{2.30}$$

if $1 \leq \ell \leq [\frac{l}{2}]$, by Hölder's inequality and Lemma A.1, we have

$$\begin{aligned}
\|\nabla^\ell u_1 \cdot \nabla \nabla^{l-\ell} n_1\|_{L^2} & \lesssim \|\nabla^{l+1-\ell} n_1\|_{L^6} \|\nabla^\ell u_1\|_{L^3} \\
& \lesssim \|\nabla n_1\|_{L^2}^{\frac{\ell-1}{2}} \|\nabla^{l+1} n_1\|_{L^2}^{\frac{l-\ell+1}{2}} \|\nabla^\alpha u_1\|_{L^2}^{\frac{l-\ell+1}{2}} \|\nabla^{l+1} u_1\|_{L^2}^{\frac{\ell-1}{2}} \\
& \lesssim (\|\nabla n_1\|_{L^2} + \|u_1\|_{H^3}) \|\nabla^{l+1}(n_1, u_1)\|_{L^2}, \tag{2.31}
\end{aligned}$$

where α is defined by

$$\ell + \frac{1}{2} = \alpha \times \frac{l - \ell + 1}{l} + (l + 1) \times \frac{\ell - 1}{l} \implies \alpha = \frac{3l - 2\ell + 2}{2l - 2\ell + 2} \in \left[\frac{3}{2}, 3\right) \text{ since } \ell \leq \frac{l}{2};$$

if $\lfloor \frac{l}{2} \rfloor + 1 \leq \ell \leq l$, by Hölder's inequality and Lemma A.1 again, we have

$$\begin{aligned} \|\nabla^\ell u_1 \cdot \nabla \nabla^{l-\ell} n_1\|_{L^2} &\lesssim \|\nabla^{l+1-\ell} n_1\|_{L^3} \|\nabla^\ell u_1\|_{L^6} \\ &\lesssim \|\nabla^\alpha n_1\|_{L^2}^{\frac{\ell+1}{l+1}} \|\nabla^{l+1} n_1\|_{L^2}^{\frac{l-\ell}{l+1}} \|u_1\|_{L^2}^{\frac{l-\ell}{l+1}} \|\nabla^{l+1} u_1\|_{L^2}^{\frac{\ell+1}{l+1}} \\ &\lesssim (\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}) \|\nabla^{l+1}(n_1, u_1)\|_{L^2}, \end{aligned} \quad (2.32)$$

where α is defined by

$$l - \ell + \frac{3}{2} = \alpha \times \frac{\ell + 1}{l + 1} + l - \ell \implies \alpha = \frac{3l + 3}{2\ell + 2} \in \left[\frac{3}{2}, 3\right) \text{ since } \ell \geq \frac{l + 1}{2}.$$

Hence, by (2.30)–(2.32), we have

$$\|\nabla^l(u_1 \cdot \nabla n_1)\|_{L^2} \lesssim (\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}) \|\nabla^{l+1}(n_1, u_1)\|_{L^2}. \quad (2.33)$$

Similarly, we also have

$$\|\nabla^l(u_2 \cdot \nabla n_1, n_1 \operatorname{div} u_1, n_1 \operatorname{div} u_2)\|_{L^2} \lesssim (\|\nabla n_1\|_{H^2} + \|(u_1, u_2)\|_{H^3}) \|\nabla^{l+1}(n_1, u_1, u_2)\|_{L^2}. \quad (2.34)$$

Using the commutator estimate of Lemma 2.3, we have

$$\begin{aligned} \|\nabla^l(u_1 \cdot \nabla n_2)\|_{L^2} &\leq \|u_1 \cdot \nabla^l \nabla n_2\|_{L^2} + \|[\nabla^l, u_1] \cdot \nabla n_2\|_{L^2} \\ &\leq \|u_1\|_{L^\infty} \|\nabla^{l+1} n_2\|_{L^2} + C_l \|\nabla u_1\|_{L^\infty} \|\nabla^l n_2\|_{L^2} + C_l \|\nabla^l u_1\|_{L^2} \|\nabla n_2\|_{L^\infty} \\ &\leq C_l \|(n_2, u_1)\|_{H^3} (\|\nabla^l(n_2, u_1)\|_{L^2} + \|\nabla^{l+1} n_2\|_{L^2}). \end{aligned} \quad (2.35)$$

Similarly,

$$\begin{aligned} \|\nabla^l(u_2 \cdot \nabla n_2, n_2 \operatorname{div} u_2, n_2 \operatorname{div} u_1)\|_{L^2} \\ \leq C_l \|(n_2, u_1, u_2)\|_{H^3} (\|\nabla^l(n_2, u_1, u_2)\|_{L^2} + \|\nabla^{l+1}(n_2, u_1, u_2)\|_{L^2}). \end{aligned} \quad (2.36)$$

Hence, we obtain

$$\begin{aligned} &\int \nabla^l \partial_t u_1 \cdot \nabla \nabla^l n_1 + \nabla^l \partial_t u_2 \cdot \nabla \nabla^l n_2 \\ &\geq \frac{d}{dt} \int \nabla^l u_1 \cdot \nabla^l \nabla n_1 + \nabla^l u_2 \cdot \nabla^l \nabla n_2 - C \|\nabla^{l+1}(u_1, u_2)\|_{L^2}^2 \\ &\quad - C_l \left(\|\nabla n_1\|_{H^2}^2 + \|(n_2, u_1, u_2)\|_{H^3}^2 \right) \left(\|\nabla^l(n_2, u_1, u_2)\|_{L^2}^2 + \|\nabla^{l+1}(n_1, n_2, u_1, u_2)\|_{L^2}^2 \right). \end{aligned} \quad (2.37)$$

Next, integrating by parts and using the equation (1.5)₇, we have

$$\begin{aligned} &2\nu \int \nabla^l E \cdot \nabla \nabla^l n_2 \\ &= -2\nu \int \nabla^l \operatorname{div} E \nabla^l n_2 = -2\nu^2 \int \nabla^l \left(f\left(\frac{n_1 + n_2}{2}\right) - f\left(\frac{n_1 - n_2}{2}\right) \right) \nabla^l n_2 \\ &= -2\nu^2 \int \nabla^l \left[n_2 + f\left(\frac{n_1 + n_2}{2}\right) - f\left(\frac{n_1 - n_2}{2}\right) - n_2 \right] \nabla^l n_2 \\ &\lesssim -\|\nabla^l n_2\|_{L^2}^2 + (\|\nabla n_1\|_{H^2} + \|n_2\|_{H^3}) (\|\nabla^{l+1}(n_1, n_2)\|_{L^2} + \|\nabla^l n_2\|_{L^2}). \end{aligned} \quad (2.38)$$

Here we have used the estimate

$$\begin{aligned} &\left\| \nabla^l \left[f\left(\frac{n_1 + n_2}{2}\right) - f\left(\frac{n_1 - n_2}{2}\right) - n_2 \right] \right\|_{L^2} \\ &\lesssim (\|\nabla n_1\|_{H^2} + \|n_2\|_{H^3}) (\|\nabla^{l+1}(n_1, n_2)\|_{L^2} + \|\nabla^l n_2\|_{L^2}). \end{aligned} \quad (2.39)$$

In fact, by noticing that $f(\frac{n_1+n_2}{2}) - f(\frac{n_1-n_2}{2}) - n_2 \sim n_1 n_2$ and Lemma 2.2, we have

$$\left\| \nabla^l \left[f\left(\frac{n_1 + n_2}{2}\right) - f\left(\frac{n_1 - n_2}{2}\right) - n_2 \right] \right\|_{L^2} \lesssim \|\nabla^l(n_1 n_2)\|_{L^2} \lesssim \sum_{0 \leq \ell \leq l} \|\nabla^\ell n_1 \nabla^{l-\ell} n_2\|_{L^2}. \quad (2.40)$$

If $\ell = 0$, then

$$\|n_1 \nabla^l n_2\|_{L^2} \lesssim \|n_1\|_{L^6} \|\nabla^l n_2\|_{L^3} \lesssim \|\nabla n_1\|_{L^2} \|\nabla^l n_2\|_{H^1}; \quad (2.41)$$

if $1 \leq \ell \leq [\frac{l}{2}]$, by Hölder's inequality and Lemma A.1, we have

$$\begin{aligned} \|\nabla^\ell n_1 \nabla^{l-\ell} n_2\|_{L^2} &\lesssim \|\nabla^\ell n_1\|_{L^3} \|\nabla^{l-\ell} n_2\|_{L^6} \\ &\lesssim \|\nabla^\alpha n_1\|_{L^2}^{\frac{l-\ell+1}{l}} \|\nabla^{l+1} n_1\|_{L^2}^{\frac{\ell-1}{l}} \|n_2\|_{L^2}^{\frac{\ell-1}{l}} \|\nabla^l n_2\|_{L^2}^{\frac{l-\ell+1}{l}} \\ &\lesssim (\|\nabla n_1\|_{H^2} + \|n_2\|_{L^2}) (\|\nabla^{l+1} n_1\|_{L^2} + \|\nabla^l n_2\|_{L^2}), \end{aligned} \quad (2.42)$$

where α is defined by

$$\ell + \frac{1}{2} = \alpha \times \frac{l-\ell+1}{l} + (l+1) \times \frac{\ell-1}{l} \implies \alpha = \frac{3l-2\ell+2}{2l-2\ell+2} \in \left[\frac{3}{2}, 3\right) \text{ since } \ell \leq \frac{l}{2};$$

if $[\frac{l}{2}] + 1 \leq \ell \leq l$, by Hölder's inequality and Lemma A.1 again, we have

$$\begin{aligned} \|\nabla^\ell n_1 \nabla^{l-\ell} n_2\|_{L^2} &\lesssim \|\nabla^\ell n_1\|_{L^6} \|\nabla^{l-\ell} n_2\|_{L^3} \\ &\lesssim \|\nabla n_1\|_{L^2}^{\frac{l-\ell}{l}} \|\nabla^{l+1} n_1\|_{L^2}^{\frac{\ell}{l}} \|\nabla^\alpha n_2\|_{L^2}^{\frac{\ell}{l}} \|\nabla^l n_2\|_{L^2}^{\frac{l-\ell}{l}} \\ &\lesssim (\|\nabla n_1\|_{L^2} + \|n_2\|_{H^3}) (\|\nabla^{l+1} n_1\|_{L^2} + \|\nabla^l n_2\|_{L^2}), \end{aligned} \quad (2.43)$$

where α is defined by

$$l - \ell + \frac{1}{2} = \alpha \times \frac{\ell}{l} + l - \ell \implies \alpha = \frac{l}{2\ell} \in \left[\frac{1}{2}, 3\right) \text{ since } \ell \geq \frac{l+1}{2}.$$

By (2.40)–(2.43), we complete the proof of (2.39).

Lastly, as in (2.33)–(2.36), we have

$$\begin{aligned} &\|\nabla^l (u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 + n_1 \nabla n_2 + n_2 \nabla n_1)\|_{L^2} \\ &\quad + \|\nabla^l (u_1 \cdot \nabla u_1 + u_2 \cdot \nabla u_2 + n_1 \nabla n_1 + n_2 \nabla n_2)\|_{L^2} \\ &\leq C_l (\|\nabla n_1\|_{H^2} + \|(n_2, u_1, u_2)\|_{H^3}) (\|\nabla^l (n_2, u_1, u_2)\|_{L^2} + \|\nabla^{l+1} (n_1, n_2, u_1, u_2)\|_{L^2}). \end{aligned} \quad (2.44)$$

From the estimate of I_2 in Lemma 2.8, we have that for $l = k$ or $k+1$,

$$\begin{aligned} &\|\nabla^l (u_1 \times B, u_2 \times B)\|_{L^2} \\ &\leq C_k \left(\|(u_1, u_2)\|_{H^{\frac{k}{2}+1} \cap H^2} + \|\nabla B\|_{L^2} \right) (\|\nabla^l (u_1, u_2)\|_{L^2} + \|\nabla^{k+1} B\|_{L^2}). \end{aligned} \quad (2.45)$$

Plugging these estimates (2.37), (2.38), (2.44) and (2.45) into (2.29), by Cauchy's inequality, we obtain

$$\begin{aligned} &\frac{d}{dt} \sum_{l=k}^{k+1} \int \nabla^l u_1 \cdot \nabla \nabla^l n_1 + \nabla^l u_2 \cdot \nabla \nabla^l n_2 + \lambda \left(\sum_{l=k+1}^{k+2} \|\nabla^l n_1\|_{L^2}^2 + \sum_{l=k}^{k+2} \|\nabla^l n_2\|_{L^2}^2 \right) \\ &\leq C \sum_{l=k}^{k+2} \|\nabla^l (u_1, u_2)\|_{L^2}^2 \\ &\quad + C_k G \left(\sum_{l=k+1}^{k+2} \|\nabla^l n_1\|_{L^2}^2 + \sum_{l=k}^{k+2} \|\nabla^l (n_2, u_1, u_2)\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 \right). \end{aligned} \quad (2.46)$$

Here G is well-defined above. This completes the dissipative estimates for n_1, n_2 .

Step 2: Dissipative estimate of E .

Applying ∇^l ($l = k, k+1$) to (1.5)₄ and then taking the L^2 inner product with $-\nabla^l E$, we obtain

$$\begin{aligned} &-\int \nabla^l \partial_t u_2 \cdot \nabla^l E + 2\nu \|\nabla^l E\|_{L^2}^2 \leq \int \nabla \nabla^l n_2 \cdot \nabla^l E + C \|\nabla^l (u_1, u_2)\|_{L^2} \|\nabla^l E\|_{L^2} \\ &\quad + \|\nabla^l (u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 + n_1 \nabla n_2 + n_2 \nabla n_1 + u_1 \times B)\|_{L^2} \|\nabla^l E\|_{L^2}. \end{aligned} \quad (2.47)$$

Again, the delicate first term on the left-hand side of (2.47) involves $\partial_t \nabla^l u_2$, and the key idea is to integrate by parts in the t -variable and use the equation (1.5)₅ in the Maxwell system. Thus we obtain

$$\begin{aligned} &-\int \nabla^l \partial_t u_2 \cdot \nabla^l E \\ &= -\frac{d}{dt} \int \nabla^l u_2 \cdot \nabla^l E + \int \nabla^l u_2 \cdot \nabla^l \partial_t E \\ &= -\frac{d}{dt} \int \nabla^l u_2 \cdot \nabla^l E - \nu \|\nabla^l u_2\|_{L^2}^2 + \int \nabla^l u_2 \cdot \nabla^l (g_5 + \nu \nabla \times B). \end{aligned} \quad (2.48)$$

From the estimates of I_3 in Lemma 2.8, we have that

$$\|\nabla^l g_5\|_{L^2} \leq C_l (\|\nabla n_1\|_{L^2} + \|(n_2, u_1, u_2)\|_{H^l \cap H^1}) (\|\nabla^{l+1}(n_1, n_2)\|_{L^2} + \|\nabla^l(u_1, u_2)\|_{L^2}).$$

We must be much more careful about the remaining term in (2.48) since there is no small factor in front of it. The key is to use Cauchy's inequality and distinct the cases of $l = k$ and $l = k + 1$ due to the weakest dissipative estimate of B . For $l = k$, we have

$$\nu \int \nabla^k u_2 \cdot \nabla \times \nabla^k B \leq \varepsilon \|\nabla^{k+1} B\|_{L^2}^2 + C_\varepsilon \|\nabla^k u_2\|_{L^2}^2;$$

for $l = k + 1$, integrating by parts, we obtain

$$\begin{aligned} \nu \int \nabla^{k+1} u_2 \cdot \nabla \times \nabla^{k+1} B &= \nu \int \nabla \times \nabla^{k+1} u_2 \cdot \nabla^{k+1} B \\ &\leq \varepsilon \|\nabla^{k+1} B\|_{L^2}^2 + C_\varepsilon \|\nabla^{k+2} u_2\|_{L^2}^2. \end{aligned} \quad (2.49)$$

Plugging these estimates (2.48)–(2.49) and (2.38), (2.44) and (2.45) from Step 1 into (2.47), by Cauchy's inequality, we then obtain

$$\begin{aligned} -\frac{d}{dt} \sum_{l=k}^{k+1} \int \nabla^l u_2 \cdot \nabla^l E + \lambda \sum_{l=k}^{k+1} \|\nabla^l E\|_{L^2}^2 &\leq \varepsilon \|\nabla^{k+1} B\|_{L^2}^2 + C_\varepsilon \sum_{l=k}^{k+2} \|\nabla^l(u_1, u_2)\|_{L^2}^2 \\ &\quad + C_k G \left(\sum_{l=k+1}^{k+2} \|\nabla^l n_1\|_{L^2}^2 + \sum_{l=k}^{k+2} \|\nabla^l(n_2, u_1, u_2)\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 \right). \end{aligned} \quad (2.50)$$

This completes the dissipative estimate for E .

Step 3: Dissipative estimate of B .

Applying ∇^k to (1.5)₅ and then taking the L^2 inner product with $-\nabla \times \nabla^k B$, we obtain

$$\begin{aligned} &-\int \nabla^k \partial_t E \cdot \nabla \times \nabla^k B + \nu \|\nabla \times \nabla^k B\|_{L^2}^2 \\ &\leq \nu \|\nabla^k u_2\|_{L^2} \|\nabla \times \nabla^k B\|_{L^2} + \|\nabla^k g_5\|_{L^2} \|\nabla \times \nabla^k B\|_{L^2}. \end{aligned} \quad (2.51)$$

Integrating by parts for both the t - and x -variables and using the equation (1.5)₆, we have

$$\begin{aligned} -\int \nabla^k \partial_t E \cdot \nabla \times \nabla^k B &= -\frac{d}{dt} \int \nabla^k E \cdot \nabla \times \nabla^k B + \int \nabla \times \nabla^k E \cdot \nabla^k \partial_t B \\ &= -\frac{d}{dt} \int \nabla^k E \cdot \nabla \times \nabla^k B - \nu \|\nabla \times \nabla^k E\|_{L^2}^2. \end{aligned}$$

From the estimates of I_3 in Lemma 2.8, we have that

$$\|\nabla^k g_5\|_{L^2} \leq C_l (\|\nabla n_1\|_{L^2} + \|(n_2, u_1, u_2)\|_{H^k \cap H^1}) (\|\nabla^{k+1}(n_1, n_2)\|_{L^2} + \|\nabla^k(u_1, u_2)\|_{L^2}).$$

Plugging the estimates above into (2.51) and by Cauchy's inequality, since $\operatorname{div} B = 0$, we then obtain

$$\begin{aligned} -\frac{d}{dt} \int \nabla^k E \cdot \nabla^k \nabla \times B + \lambda \|\nabla^{k+1} B\|_{L^2}^2 &\leq C \|\nabla^k u_2\|_{L^2}^2 + C \|\nabla^{k+1} E\|_{L^2}^2 \\ &\quad + C_k \left(\|\nabla n_1\|_{L^2}^2 + \|(n_2, u_1, u_2)\|_{H^k \cap H^1}^2 \right) \left(\|\nabla^{k+1}(n_1, n_2)\|_{L^2}^2 + \|\nabla^k(u_1, u_2)\|_{L^2}^2 \right). \end{aligned} \quad (2.52)$$

This completes the dissipative estimate for B .

Step 4: Conclusion.

Multiplying (2.52) by a small enough but fixed constant η and then adding it to (2.50) so that the second term on the right-hand side of (2.52) can be absorbed, then choosing ε small enough so that the first term on the right-hand side of (2.50) can be absorbed; we obtain

$$\begin{aligned} &\frac{d}{dt} \left(\sum_{l=k}^{k+1} \int \nabla^l u_2 \cdot \nabla^l E - \eta \int \nabla^k E \cdot \nabla^k \nabla \times B \right) + \lambda \left(\sum_{l=k}^{k+1} \|\nabla^l E\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 \right) \\ &\leq C \sum_{l=k}^{k+2} \|\nabla^l u_2\|_{L^2}^2 + C_k G \left(\sum_{l=k+1}^{k+2} \|\nabla^l n_1\|_{L^2}^2 + \sum_{l=k}^{k+2} \|\nabla^l(n_2, u_1, u_2)\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 \right). \end{aligned}$$

Adding the inequality above to (2.46), we get (2.28). \square

2.3. Negative Sobolev estimates. In this subsection, we will derive the evolution of the negative Sobolev norms of $U := (n_1, n_2, u_1, u_2, E, B)$. In order to estimate the nonlinear terms, we need to restrict ourselves to that $s \in (0, 3/2)$. We will establish the following lemma.

Lemma 2.10. *For $s \in (0, 1/2]$, we have*

$$\frac{d}{dt} \|U\|_{\dot{H}^{-s}}^2 + \lambda \|(u_1, u_2)\|_{\dot{H}^{-s}}^2 \lesssim \left(\|(n_2, u_1, u_2)\|_{H^2}^2 + \|\nabla(n_1, B)\|_{H^1}^2 \right) \|U\|_{\dot{H}^{-s}}; \quad (2.53)$$

and for $s \in (1/2, 3/2)$, we have

$$\begin{aligned} \frac{d}{dt} \|U\|_{\dot{H}^{-s}}^2 + \lambda \|(u_1, u_2)\|_{\dot{H}^{-s}}^2 &\lesssim \|(\nabla n_1, n_2, u_1, u_2)\|_{H^1}^2 \|U\|_{\dot{H}^{-s}} \\ &\quad + \|(n_1, B)\|_{L^2}^{s-1/2} \|\nabla(n_1, B)\|_{L^2}^{3/2-s} \|(\nabla n_1, \nabla n_2, u_1, u_2, \nabla u_1, \nabla u_2)\|_{L^2} \|U\|_{\dot{H}^{-s}}. \end{aligned} \quad (2.54)$$

Proof. The Λ^{-s} ($s > 0$) energy estimate of (1.5)₁–(1.5)₆ yield

$$\begin{aligned} &\frac{d}{dt} \left(\frac{1}{2} \|(n_1, n_2, u_1, u_2)\|_{\dot{H}^{-s}}^2 + \|(E, B)\|_{\dot{H}^{-s}}^2 \right) + \nu \|(u_1, u_2)\|_{\dot{H}^{-s}}^2 \\ &= \int \Lambda^{-s} g_1 \cdot \Lambda^{-s} n_1 + \int \Lambda^{-s} (g_2 + u_2 \times B) \cdot \Lambda^{-s} u_1 \\ &\quad + \int \Lambda^{-s} g_3 \cdot \Lambda^{-s} n_2 + \int \Lambda^{-s} (g_4 + u_1 \times B) \cdot \Lambda^{-s} u_2 + 2 \int \Lambda^{-s} g_5 \cdot \Lambda^{-s} E \\ &\lesssim \|g_1\|_{\dot{H}^{-s}} \|n_1\|_{\dot{H}^{-s}} + \|g_2 + u_2 \times B\|_{\dot{H}^{-s}} \|u_1\|_{\dot{H}^{-s}} \\ &\quad + \|g_3\|_{\dot{H}^{-s}} \|n_2\|_{\dot{H}^{-s}} + \|g_4 + u_1 \times B\|_{\dot{H}^{-s}} \|u_2\|_{\dot{H}^{-s}} + \|g_5\|_{\dot{H}^{-s}} \|E\|_{\dot{H}^{-s}}. \end{aligned} \quad (2.55)$$

We now restrict the value of s in order to estimate the other terms on the right-hand side of (2.55). If $s \in (0, 1/2]$, then $1/2 + s/3 < 1$ and $3/s \geq 6$. Then applying Lemma 2.4, together with Hölder's, Sobolev's and Young's inequalities, we obtain

$$\begin{aligned} \|u_1 \cdot \nabla u_2\|_{\dot{H}^{-s}} &\lesssim \|u_1 \cdot \nabla u_2\|_{L^{\frac{1}{1/2+s/3}}} \lesssim \|u_1\|_{L^{3/s}} \|\nabla u_2\|_{L^2} \\ &\lesssim \|\nabla u_1\|_{L^2}^{1/2+s} \|\nabla^2 u_1\|_{L^2}^{1/2-s} \|\nabla u_2\|_{L^2} \\ &\lesssim \|\nabla u_1\|_{H^1}^2 + \|\nabla u_2\|_{L^2}^2. \end{aligned}$$

We can similarly bound the other terms in the $g_1 \sim g_5$ and $(u_1 + u_2) \times B$. So we have

$$\sum_{i=1}^5 \|g_i\|_{\dot{H}^{-s}} + \|(u_1 + u_2) \times B\|_{\dot{H}^{-s}} \lesssim \|(n_2, u_1, u_2)\|_{H^2}^2 + \|\nabla(n_1, B)\|_{H^1}^2. \quad (2.56)$$

Now if $s \in (1/2, 3/2)$, we shall estimate the right-hand side of (2.55) in a different way. Since $s \in (1/2, 3/2)$, we have that $1/2 + s/3 < 1$ and $2 < 3/s < 6$. Then applying Lemma 2.4 and using (different) Sobolev's inequality, we have

$$\begin{aligned} \|u_1 \cdot \nabla u_2\|_{\dot{H}^{-s}} &\lesssim \|u_1\|_{L^{3/s}} \|\nabla u_2\|_{L^2} \lesssim \|u_1\|_{L^2}^{s-1/2} \|\nabla u_1\|_{L^2}^{3/2-s} \|\nabla u_2\|_{L^2} \\ &\lesssim \|u_1\|_{H^1}^2 + \|\nabla u_2\|_{L^2}^2. \end{aligned}$$

In particular, we must be careful about the terms involved with n_1 and B since they are both degenerately dissipative. For example,

$$\begin{aligned} \|n_1 \nabla n_2\|_{\dot{H}^{-s}} &\lesssim \|n_1\|_{L^2}^{s-1/2} \|\nabla n_1\|_{L^2}^{3/2-s} \|\nabla n_2\|_{L^2}; \\ \|u_2 \times B\|_{\dot{H}^{-s}} &\lesssim \|B\|_{L^2}^{s-1/2} \|\nabla B\|_{L^2}^{3/2-s} \|u_2\|_{L^2}. \end{aligned}$$

Then, we have

$$\begin{aligned} \sum_{i=1}^5 \|g_i\|_{\dot{H}^{-s}} + \|(u_1 + u_2) \times B\|_{\dot{H}^{-s}} &\lesssim \|(\nabla n_1, n_2, u_1, u_2)\|_{H^1}^2 \\ &\quad + \|(n_1, B)\|_{L^2}^{s-1/2} \|\nabla(n_1, B)\|_{L^2}^{3/2-s} \|(\nabla n_1, \nabla n_2, u_1, u_2, \nabla u_1, \nabla u_2)\|_{L^2}. \end{aligned} \quad (2.57)$$

Hence, we deduce (2.53) from (2.56) and (2.54) from (2.57). \square

2.4. Negative Besov estimates. In this subsection, we will derive the evolution of the negative Besov norms of $U := (n_1, n_2, u_1, u_2, E, B)$. The argument is similar to the previous subsection.

Lemma 2.11. *For $s \in (0, 1/2]$, we have*

$$\frac{d}{dt} \|U\|_{\dot{B}_{2,\infty}^{-s}}^2 + \lambda \|(u_1, u_2)\|_{\dot{B}_{2,\infty}^{-s}}^2 \lesssim \left(\|(n_2, u_1, u_2)\|_{H^2}^2 + \|\nabla(n_1, B)\|_{H^1}^2 \right) \|U\|_{\dot{B}_{2,\infty}^{-s}}^2;$$

and for $s \in (1/2, 3/2]$, we have

$$\begin{aligned} \frac{d}{dt} \|U\|_{\dot{B}_{2,\infty}^{-s}}^2 + \lambda \|(u_1, u_2)\|_{\dot{B}_{2,\infty}^{-s}}^2 &\lesssim \|(\nabla n_1, n_2, u_1, u_2)\|_{H^1}^2 \|U\|_{\dot{B}_{2,\infty}^{-s}}^2 \\ &\quad + \|(n_1, B)\|_{L^2}^{s-1/2} \|\nabla(n_1, B)\|_{L^2}^{3/2-s} \|(\nabla n_1, \nabla n_2, u_1, u_2, \nabla u_1, \nabla u_2)\|_{L^2} \|U\|_{\dot{B}_{2,\infty}^{-s}}. \end{aligned}$$

Proof. The $\dot{\Delta}_j$ energy estimates of (1.5)₁–(1.5)₆ yield, with multiplication of 2^{-2sj} and then taking the supremum over $j \in \mathbb{Z}$,

$$\begin{aligned} &\frac{d}{dt} \left(\frac{1}{2} \|(n_1, n_2, u_1, u_2)\|_{\dot{B}_{2,\infty}^{-s}}^2 + \|(E, B)\|_{\dot{B}_{2,\infty}^{-s}}^2 \right) + \nu \|(u_1, u_2)\|_{\dot{B}_{2,\infty}^{-s}}^2 \\ &\lesssim \sup_{j \in \mathbb{Z}} 2^{-2sj} \int \dot{\Delta}_j g_1 \cdot \dot{\Delta}_j n_1 + \dot{\Delta}_j (g_2 + u_2 \times B) \cdot \dot{\Delta}_j u_1 \\ &\quad + \sup_{j \in \mathbb{Z}} 2^{-2sj} \int \dot{\Delta}_j g_3 \cdot \dot{\Delta}_j n_2 + \dot{\Delta}_j (g_4 + u_1 \times B) \cdot \dot{\Delta}_j u_2 + 2 \dot{\Delta}_j g_5 \cdot \dot{\Delta}_j E \\ &\lesssim \|g_1\|_{\dot{B}_{2,\infty}^{-s}} \|n_1\|_{\dot{B}_{2,\infty}^{-s}} + \|g_2 + u_2 \times B\|_{\dot{B}_{2,\infty}^{-s}} \|u_1\|_{\dot{B}_{2,\infty}^{-s}} \\ &\quad + \|g_3\|_{\dot{B}_{2,\infty}^{-s}} \|n_2\|_{\dot{B}_{2,\infty}^{-s}} + \|g_4 + u_1 \times B\|_{\dot{B}_{2,\infty}^{-s}} \|u_2\|_{\dot{B}_{2,\infty}^{-s}} + \|g_5\|_{\dot{B}_{2,\infty}^{-s}} \|E\|_{\dot{B}_{2,\infty}^{-s}}. \end{aligned}$$

Then the proof is exactly the same as the proof of Lemma 2.10 except that we should apply Lemma 2.5 instead to estimate the $\dot{B}_{2,\infty}^{-s}$ norm. Note that we allow $s = 3/2$. \square

3. PROOF OF THEOREMS

3.1. Proof of Theorem 1.1. In this subsection, we will prove the unique global solution to the system (1.5), and the key point is that we only assume the H^3 norm of initial data is small.

Step 1. Global small \mathcal{E}_3 solution.

We first close the energy estimates at the H^3 level by assuming a priori that $\sqrt{\mathcal{E}_3(t)} \leq \delta$ is sufficiently small. Taking $k = 0, 1$ in (2.3) of Lemma 2.8 and then summing up, we obtain

$$\frac{d}{dt} \sum_{l=0}^3 \|\nabla^l U\|_{L^2}^2 + \lambda \sum_{l=0}^3 \|\nabla^l (u_1, u_2)\|_{L^2}^2 \lesssim \sqrt{\mathcal{E}_3} \mathcal{D}_3 + \sqrt{\mathcal{D}_3} \sqrt{\mathcal{D}_3} \sqrt{\mathcal{E}_3} \lesssim \delta \mathcal{D}_3. \quad (3.1)$$

Taking $k = 0, 1$ in (2.28) of Lemma 2.9 and then summing up, we obtain

$$\begin{aligned} &\frac{d}{dt} \left(\sum_{l=0}^2 \int \nabla^l u_1 \cdot \nabla \nabla^l n_1 + \nabla^l u_2 \cdot \nabla \nabla^l n_2 + \sum_{l=0}^2 \int \nabla^l u_2 \cdot \nabla^l E - \eta \sum_{l=0}^1 \int \nabla^l E \cdot \nabla^l \nabla \times B \right) \\ &\quad + \lambda \left(\sum_{l=1}^3 \|\nabla^l n_1\|_{L^2}^2 + \sum_{l=0}^3 \|\nabla^l n_2\|_{L^2}^2 + \sum_{l=0}^2 \|\nabla^l E\|_{L^2}^2 + \sum_{l=1}^2 \|\nabla^l B\|_{L^2}^2 \right) \\ &\lesssim \sum_{l=0}^3 \|\nabla^l (u_1, u_2)\|_{L^2}^2 + \delta^2 \mathcal{D}_3. \end{aligned} \quad (3.2)$$

Since δ is small, we deduce from (3.2) $\times \varepsilon$ + (3.1) that there exists an instant energy functional $\tilde{\mathcal{E}}_3$ equivalent to \mathcal{E}_3 such that

$$\frac{d}{dt} \tilde{\mathcal{E}}_3 + \mathcal{D}_3 \leq 0.$$

Integrating the inequality above directly in time, we obtain (1.7). By a standard continuity argument, we then close the a priori estimates if we assume at initial time that $\mathcal{E}_3(0) \leq \delta_0$ is sufficiently small. This concludes the unique global small \mathcal{E}_3 solution.

Step 2. Global \mathcal{E}_N solution.

We shall prove this by an induction on $N \geq 3$. By (1.7), then (1.8) is valid for $N = 3$. Assume (1.8) holds for $N - 1$ (then now $N \geq 4$). Taking $k = 0, \dots, N - 2$ in (2.3) of Lemma 2.8 and then summing up, we obtain

$$\begin{aligned} & \frac{d}{dt} \sum_{l=0}^N \|\nabla^l U\|_{L^2}^2 + \lambda \sum_{l=0}^N \|\nabla^l(u_1, u_2)\|_{L^2}^2 \\ & \leq C_N \sqrt{\mathcal{D}_{N-1}} \sqrt{\mathcal{E}_N} \sqrt{\mathcal{D}_N} + C \sqrt{\mathcal{D}_{N-1}} \sqrt{\mathcal{D}_N} \sqrt{\mathcal{E}_N} \leq C_N \sqrt{\mathcal{D}_{N-1}} \sqrt{\mathcal{E}_N} \sqrt{\mathcal{D}_N}. \end{aligned} \quad (3.3)$$

Here we have used the fact that $3 \leq \frac{N-2}{2} + 2 \leq N - 2 + 1$ since $N \geq 4$. Note that it is important that we have put the two first factors in (2.3) into the dissipation.

Taking $k = 0, \dots, N - 2$ in (2.28) of Lemma 2.9 and then summing up, we obtain

$$\begin{aligned} & \frac{d}{dt} \left(\sum_{l=0}^{N-1} \int \nabla^l u_1 \cdot \nabla \nabla^l n_1 + \nabla^l u_2 \cdot \nabla \nabla^l n_2 + \sum_{l=0}^{N-1} \int \nabla^l u_2 \cdot \nabla^l E - \eta \sum_{l=0}^{N-2} \int \nabla^l E \cdot \nabla \times \nabla^l B \right) \\ & + \lambda \left(\sum_{l=1}^N \|\nabla^l n_1\|_{L^2}^2 + \sum_{l=0}^N \|\nabla^l n_2\|_{L^2}^2 + \sum_{l=0}^{N-1} \|\nabla^l E\|_{L^2}^2 + \sum_{l=1}^{N-1} \|\nabla^l B\|_{L^2}^2 \right) \\ & \leq C \sum_{l=0}^N \|\nabla^l(u_1, u_2)\|_{L^2}^2 + C_N \sqrt{\mathcal{D}_{N-1}} \sqrt{\mathcal{D}_N} \sqrt{\mathcal{E}_N}. \end{aligned} \quad (3.4)$$

We deduce from (3.4) $\times \varepsilon + (3.3)$ that there exists an instant energy functional $\tilde{\mathcal{E}}_N$ equivalent to \mathcal{E}_N such that, by Cauchy's inequality,

$$\frac{d}{dt} \tilde{\mathcal{E}}_N + \mathcal{D}_N \leq C_N \sqrt{\mathcal{D}_{N-1}} \sqrt{\mathcal{E}_N} \sqrt{\mathcal{D}_N} \leq \varepsilon \mathcal{D}_N + C_{N,\varepsilon} \mathcal{D}_{N-1} \mathcal{E}_N.$$

This implies

$$\frac{d}{dt} \tilde{\mathcal{E}}_N + \frac{1}{2} \mathcal{D}_N \leq C_N \mathcal{D}_{N-1} \mathcal{E}_N.$$

We then use the standard Gronwall lemma and the induction hypothesis to deduce that

$$\begin{aligned} \mathcal{E}_N(t) + \int_0^t \mathcal{D}_N(\tau) d\tau & \leq C \mathcal{E}_N(0) e^{C_N \int_0^t \mathcal{D}_{N-1}(\tau) d\tau} \\ & \leq C \mathcal{E}_N(0) e^{C_N P_{N-1}(\mathcal{E}_{N-1}(0))} \\ & \leq C \mathcal{E}_N(0) e^{C_N P_{N-1}(\mathcal{E}_N(0))} \equiv P_N(\mathcal{E}_N(0)). \end{aligned}$$

This concludes the global \mathcal{E}_N solution. The proof of Theorem 1.1 is completed. \square

3.2. Proof of Theorem 1.2. In this subsection, we will prove the various time decay rates of the unique global solution to the system (1.5) obtained in Theorem 1.1. Fix $N \geq 5$. We need to assume that $\mathcal{E}_N(0) \leq \delta_0 = \delta_0(N)$ is small. Then Theorem 1.1 implies that there exists a unique global \mathcal{E}_N solution, and $\mathcal{E}_N(t) \leq P_N(\mathcal{E}_N(0)) \leq \delta_0$ is small for all time t . Since now our δ_0 is relative small with respect to N , we just ignore the N dependence of the constants in the energy estimates in the previous section.

Step 1. Basic decay.

For the convenience of presentations, we define a family of energy functionals and the corresponding dissipation rates with *minimum derivative counts* as

$$\mathcal{E}_k^{k+2} = \sum_{l=k}^{k+2} \|\nabla^l U\|_{L^2}^2 \quad (3.5)$$

and

$$\mathcal{D}_k^{k+2} = \sum_{l=k+1}^{k+2} \|\nabla^l n_1\|_{L^2}^2 + \sum_{l=k}^{k+2} \|\nabla^l(n_2, u_1, u_2)\|_{L^2}^2 + \sum_{l=k}^{k+1} \|\nabla^l E\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2. \quad (3.6)$$

By Lemma 2.8, we have that for $k = 0, \dots, N - 2$,

$$\begin{aligned} & \frac{d}{dt} \sum_{l=k}^{k+2} \|\nabla^l U\|_{L^2}^2 + \lambda \sum_{l=k}^{k+2} \|\nabla^l(u_1, u_2)\|_{L^2}^2 \\ & \lesssim \sqrt{\delta_0} \mathcal{D}_k^{k+2} + \|(n_2, u_1, u_2)\|_{L^\infty} \|\nabla^{k+2}(n_1, n_2, u_1, u_2)\|_{L^2} \|\nabla^{k+2}(E, B)\|_{L^2}. \end{aligned} \quad (3.7)$$

By Lemma 2.9, we have that for $k = 0, \dots, N-2$,

$$\begin{aligned} & \frac{d}{dt} \left(\sum_{l=k}^{k+1} \int \nabla^l u_1 \cdot \nabla \nabla^l n_1 + \nabla^l u_2 \cdot \nabla \nabla^l n_2 + \sum_{l=k}^{k+1} \int \nabla^l u_2 \cdot \nabla^l E - \eta \int \nabla^k E \cdot \nabla^k \nabla \times B \right) \\ & + \lambda \left(\sum_{l=k+1}^{k+2} \|\nabla^l n_1\|_{L^2}^2 + \sum_{l=k}^{k+2} \|\nabla^l n_2\|_{L^2}^2 + \sum_{l=k}^{k+1} \|\nabla^l E\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 \right) \\ & \lesssim \sum_{l=k}^{k+2} \|\nabla^l(u_1, u_2)\|_{L^2}^2 + \delta_0 \sum_{l=k}^{k+2} \|\nabla^l(u_1, u_2)\|_{L^2}^2. \end{aligned} \quad (3.8)$$

Since δ_0 is small, we deduce from (3.8) $\times \varepsilon + (3.7)$ that there exists an instant energy functional $\tilde{\mathcal{E}}_k^{k+2}$ equivalent to \mathcal{E}_k^{k+2} such that

$$\frac{d}{dt} \tilde{\mathcal{E}}_k^{k+2} + \mathcal{D}_k^{k+2} \lesssim \|(n_2, u_1, u_2)\|_{L^\infty} \|\nabla^{k+2}(n_1, n_2, u_1, u_2)\|_{L^2} \|\nabla^{k+2}(E, B)\|_{L^2}. \quad (3.9)$$

Note that we can not absorb the right-hand side of (3.9) by the dissipation \mathcal{D}_k^{k+2} since it does not contain $\|\nabla^{k+2}(E, B)\|_{L^2}^2$. We will distinct the arguments by the value of k . If $k = 0$ or $k = 1$, we bound $\|\nabla^{k+2}(E, B)\|_{L^2}$ by the energy. Then we have that for $k = 0, 1$,

$$\frac{d}{dt} \tilde{\mathcal{E}}_k^{k+2} + \mathcal{D}_k^{k+2} \lesssim \sqrt{\mathcal{D}_k^{k+2}} \sqrt{\mathcal{D}_k^{k+2}} \sqrt{\mathcal{E}_3} \lesssim \sqrt{\delta_0} \mathcal{D}_k^{k+2},$$

which implies

$$\frac{d}{dt} \tilde{\mathcal{E}}_k^{k+2} + \mathcal{D}_k^{k+2} \leq 0.$$

If $k \geq 2$, we have to bound $\|\nabla^{k+2}(E, B)\|_{L^2}$ in term of $\|\nabla^{k+1}(E, B)\|_{L^2}$ since $\sqrt{\mathcal{D}_k^{k+2}}$ can not control $\|(n_2, u_1, u_2)\|_{L^\infty}$. The key point is to use the regularity interpolation method developed in [12, 23]. By Lemma 2.1, we have

$$\begin{aligned} & \|(n_2, u_1, u_2)\|_{L^\infty} \|\nabla^{k+2}(n_1, n_2, u_1, u_2)\|_{L^2} \|\nabla^{k+2}(E, B)\|_{L^2} \\ & \lesssim \|(n_2, u_1, u_2)\|_{L^2}^{1-\frac{3}{2k}} \|\nabla^k(n_2, u_1, u_2)\|_{L^2}^{\frac{3}{2k}} \|\nabla^{k+2}(n_1, n_2, u_1, u_2)\|_{L^2} \\ & \quad \cdot \|\nabla^{k+1}(E, B)\|_{L^2}^{1-\frac{3}{2k}} \|\nabla^\alpha(E, B)\|_{L^2}^{\frac{3}{2k}}, \end{aligned} \quad (3.10)$$

where α is defined by

$$k+2 = (k+1) \times \left(1 - \frac{3}{2k}\right) + \alpha \times \frac{3}{2k} \implies \alpha = \frac{5}{3}k + 1.$$

Hence, for $k \geq 2$, if $N \geq \frac{5}{3}k + 1 \iff 2 \leq k \leq \frac{3}{5}(N-1)$, then by (3.10), we deduce from (3.9) that

$$\frac{d}{dt} \tilde{\mathcal{E}}_k^{k+2} + \mathcal{D}_k^{k+2} \lesssim \sqrt{\mathcal{E}_N} \mathcal{D}_k^{k+2} \lesssim \sqrt{\delta_0} \mathcal{D}_k^{k+2},$$

which allow us to arrive at that for any integer k with $0 \leq k \leq \frac{3}{5}(N-1)$ (note that $N-2 \geq \frac{3}{5}(N-1) \geq 2$ since $N \geq 5$), we have

$$\frac{d}{dt} \tilde{\mathcal{E}}_k^{k+2} + \mathcal{D}_k^{k+2} \leq 0. \quad (3.11)$$

We now begin to derive the decay rate from (3.11). In fact, we have proved (1.9) or (1.10) in the similar fashion of [24] by utilizing Lemma 2.10 and 2.11. Using Lemma 2.6, we have that for $s \geq 0$ and $k+s > 0$,

$$\|\nabla^k(n_1, B)\|_{L^2} \leq \|(n_1, B)\|_{\dot{H}^{-s}}^{\frac{1}{k+1+s}} \|\nabla^{k+1}(n_1, B)\|_{L^2}^{\frac{k+s}{k+1+s}} \leq C_0 \|\nabla^{k+1}(n_1, B)\|_{L^2}^{\frac{k+s}{k+1+s}}.$$

Similarly, using Lemma 2.7, we have that for $s > 0$ and $k+s > 0$,

$$\|\nabla^k(n_1, B)\|_{L^2} \leq \|(n_1, B)\|_{\dot{B}_{2,\infty}^{-s}}^{\frac{1}{k+1+s}} \|\nabla^{k+1}(n_1, B)\|_{L^2}^{\frac{k+s}{k+1+s}} \leq C_0 \|\nabla^{k+1}(n_1, B)\|_{L^2}^{\frac{k+s}{k+1+s}}.$$

On the other hand, for $k+2 < N$, we have

$$\|\nabla^{k+2}(E, B)\|_{L^2} \leq \|\nabla^{k+1}(E, B)\|_{L^2}^{\frac{N-k-2}{N-k-1}} \|\nabla^N(E, B)\|_{L^2}^{\frac{1}{N-k-1}} \leq C_0 \|\nabla^{k+1}(E, B)\|_{L^2}^{\frac{N-k-2}{N-k-1}}.$$

Then we deduce from (3.11) that

$$\frac{d}{dt} \tilde{\mathcal{E}}_k^{k+2} + \{\mathcal{E}_k^{k+2}\}^{1+\vartheta} \leq 0,$$

where $\vartheta = \max \left\{ \frac{1}{k+s}, \frac{1}{N-k-2} \right\}$. Solving this inequality directly, we obtain in particular that

$$\mathcal{E}_k^{k+2}(t) \leq \left\{ [\mathcal{E}_k^{k+2}(0)]^{-\vartheta} + \vartheta t \right\}^{-1/\vartheta} \leq C_0(1+t)^{-1/\vartheta} = C_0(1+t)^{-\min\{k+s, N-k-2\}}. \quad (3.12)$$

Notice that (3.12) holds also for $k+s=0$ or $k+2=N$. So, if we want to obtain the optimal decay rate of the whole solution for the spatial derivatives of order k , we only need to assume N large enough (for fixed k and s) so that $k+s \leq N-k-2$. Thus we should require that

$$N \geq \max \left\{ k+2, \frac{5}{3}k+1, 2k+2+s \right\} = 2k+2+s.$$

This proves the optimal decay (1.11).

Step 2. Further decay.

We first prove (1.12) and (1.13). First, noticing that $-\nu g = \operatorname{div} E$, by (1.11) and Lemma 2.2, if $N \geq 2k+4+s$, then

$$\|\nabla^k n_2(t)\|_{L^2} \lesssim \|\nabla^k g(t)\|_{L^2} \lesssim \|\nabla^{k+1} E(t)\|_{L^2} \lesssim C_0(1+t)^{-\frac{k+1+s}{2}}. \quad (3.13)$$

Next, applying ∇^k to (1.5)₂, (1.5)₄, (1.5)₅ and then multiplying the resulting identities by $\nabla^k u_1$, $\nabla^k u_2$, $\nabla^k E$ respectively, summing up and integrating over \mathbb{R}^3 , we obtain

$$\begin{aligned} & \frac{d}{dt} \int \left(\frac{1}{2} |\nabla^k(u_1, u_2)|^2 + |\nabla^k E|^2 \right) + \nu \|\nabla^k(u_1, u_2)\|_{L^2}^2 \\ &= \int \nabla^k (-\nabla n_1 + g_2 + u_2 \times B) \cdot \nabla^k u_1 + \int \nabla^k (-\nabla n_2 + g_4 + u_1 \times B) \cdot \nabla^k u_2 \\ & \quad + 2\nu \int \nabla^k (\nabla \times B + g_5) \cdot \nabla^k E \\ & \lesssim \|\nabla^{k+1} n_1\|_{L^2} \|\nabla^k u_1\|_{L^2} + \|\nabla^k (g_2 + u_2 \times B)\|_{L^2} \|\nabla^k u_1\|_{L^2} \\ & \quad + \|\nabla^{k+1} n_2\|_{L^2} \|\nabla^k u_2\|_{L^2} + \|\nabla^k (g_4 + u_1 \times B)\|_{L^2} \|\nabla^k u_2\|_{L^2} \\ & \quad + \|\nabla^k (\nabla \times B + g_5)\|_{L^2} \|\nabla^k E\|_{L^2}. \end{aligned} \quad (3.14)$$

On the other hand, taking $l=k$ in (2.47), we may have

$$\begin{aligned} - \int \nabla^k \partial_t u_2 \cdot \nabla^k E + 2\nu \|\nabla^k E\|_{L^2}^2 & \leq \int \nabla \nabla^k n_2 \cdot \nabla^k E + C \|\nabla^k(u_1, u_2)\|_{L^2} \|\nabla^k E\|_{L^2} \\ & \quad + \|\nabla^k (g_4 + u_1 \times B)\|_{L^2} \|\nabla^k E\|_{L^2}. \end{aligned} \quad (3.15)$$

Substituting (2.48) with $l=k$ into (3.15), we may then have

$$\begin{aligned} & - \frac{d}{dt} \int \nabla^k u_2 \cdot \nabla^k E + 2\nu \|\nabla^k E\|_{L^2}^2 \\ & \lesssim \|\nabla^k u_2\|_{L^2}^2 + (\|\nabla^{k+1} n_2\|_{L^2} + \|\nabla^k(u_1, u_2)\|_{L^2}) \|\nabla^k E\|_{L^2} \\ & \quad + \|\nabla^k (\nabla \times B + g_5)\|_{L^2} \|\nabla^k u_2\|_{L^2} + \|\nabla^k (g_4 + u_1 \times B)\|_{L^2} \|\nabla^k E\|_{L^2}. \end{aligned} \quad (3.16)$$

Since ε is small, we deduce from (3.16) $\times \varepsilon + (3.14)$ that there exists $\mathcal{F}_k(t) \sim \|\nabla^k(u_1, u_2, E)(t)\|_{L^2}^2$ such that, by Cauchy's inequality, Lemma 2.3, (2.22), (2.24), (2.26), (1.11) and (3.13),

$$\begin{aligned} & \frac{d}{dt} \mathcal{F}_k(t) + \mathcal{F}_k(t) \\ & \lesssim \|\nabla^{k+1}(n_1, n_2)\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 + \|\nabla^k (g_2 + u_2 \times B)\|_{L^2}^2 \\ & \quad + \|\nabla^k (g_4 + u_1 \times B)\|_{L^2}^2 + \|\nabla^k g_5\|_{L^2}^2 \\ & \lesssim \|\nabla^{k+1}(n_1, n_2, B)\|_{L^2}^2 + \left(\|(u_1, u_2)\|_{H^{\frac{k}{2}}} + \|\nabla B\|_{L^2} \right)^2 \|\nabla^{k+1} B\|_{L^2}^2 \\ & \quad + \|(n_1, n_2, u_1, u_2)\|_{L^\infty}^2 \|\nabla^{k+1}(n_1, n_2, u_1, u_2)\|_{L^2}^2 + \|\nabla n_1\|_{L^\infty}^2 \|\nabla^k n_1\|_{L^2}^2 \\ & \leq C_0(1+t)^{-(k+1+s)}, \end{aligned} \quad (3.17)$$

where we required $N \geq 2k+4+s$. Applying the standard Gronwall lemma to (3.17), we obtain

$$\mathcal{F}_k(t) \leq \mathcal{F}_k(0)e^{-t} + C_0 \int_0^t e^{-(t-\tau)} (1+\tau)^{-(k+1+s)} d\tau \lesssim C_0(1+t)^{-(k+1+s)}.$$

This implies

$$\|\nabla^k(u_1, u_2, E)(t)\|_{L^2} \lesssim \sqrt{\mathcal{F}_k(t)} \lesssim C_0(1+t)^{-\frac{k+1+s}{2}}.$$

We thus complete the proof of (1.12). Notice that (1.13) now follows by (3.13) with the improved decay rate of E in (1.12), just requiring $N \geq 2k + 6 + s$.

Now we prove (1.14). Assuming $B_\infty = 0$, then we can extract the following system from (1.5)₃–(1.5)₄, denoting $\psi = \operatorname{div} u_2$,

$$\begin{cases} \partial_t n_2 + \psi = g_3, \\ \partial_t \psi + \nu \psi - 2\nu^2 n_2 = -\Delta n_2 - \operatorname{div}(g_4 + u_1 \times B) + 2\nu^2(-g - n_2). \end{cases} \quad (3.18)$$

Here g is defined in (2.27). Applying ∇^k to (3.18) and then multiplying the resulting identities by $2\nu^2 \nabla^k n_2$, $\nabla^k \psi$, respectively, summing up and integrating over \mathbb{R}^3 , we obtain

$$\begin{aligned} & \frac{d}{dt} \int \nu^2 |\nabla^k n_2|^2 + \frac{1}{2} |\nabla^k \psi|^2 + \nu \|\nabla^k \psi\|_{L^2}^2 \\ &= 2\nu^2 \int \nabla^k g_3 \nabla^k n_2 - \int \nabla^k \Delta n_2 \nabla^k \psi \\ & \quad - \int \nabla^k [\operatorname{div}(g_4 + u_1 \times B) - 2\nu^2(-g - n_2)] \nabla^k \psi. \end{aligned} \quad (3.19)$$

Applying ∇^k to (3.18)₂ and then multiplying by $-\nabla^k n_2$, as before integrating by parts over t and x variables and using the equation (3.18)₁, we may obtain

$$\begin{aligned} & -\frac{d}{dt} \int \nabla^k \psi \nabla^k n_2 + 2\nu^2 \|\nabla^k n_2\|_{L^2}^2 = \|\nabla^k \psi\|_{L^2}^2 + \nu \int \nabla^k n_2 \nabla^k \psi - \int \nabla^k g_3 \nabla^k \psi \\ & \quad + \int \nabla^k [\Delta n_2 + \operatorname{div}(g_4 + u_1 \times B) - 2\nu^2(-g - n_2)] \nabla^k n_2. \end{aligned} \quad (3.20)$$

Since ε is small, we deduce from (3.20) $\times \varepsilon + (3.19)$ that there exists $\mathcal{G}_k(t) \sim \|\nabla^k(n_2, \psi)\|_{L^2}^2$ such that, by Cauchy's inequality,

$$\begin{aligned} \frac{d}{dt} \mathcal{G}_k(t) + \mathcal{G}_k(t) & \lesssim \|\nabla^{k+2} n_2\|_{L^2}^2 + \|\nabla^k g_3\|_{L^2}^2 + \|\nabla^{k+1} g_4\|_{L^2}^2 + \|\nabla^{k+1}(u_1 \times B)\|_{L^2}^2 \\ & \quad + \|\nabla^k(-g - n_2)\|_{L^2}^2. \end{aligned} \quad (3.21)$$

By Lemma 2.2 and Lemma 2.3, we obtain

$$\begin{aligned} \|\nabla^k(-g - n_2)\|_{L^2}^2 & \lesssim \|n_1\|_{L^\infty}^2 \|\nabla^k n_2\|_{L^2}^2 + \|\nabla^k n_1\|_{L^2}^2 \|n_2\|_{L^\infty}^2 \\ & \lesssim \delta_0 \|\nabla^k n_2\|_{L^2}^2 + \|n_2\|_{L^\infty}^2 \|\nabla^k n_1\|_{L^2}^2. \end{aligned}$$

By Lemma 2.3 and Cauchy's inequality, we obtain

$$\begin{aligned} \|\nabla^{k+1}(u_1 \times B)\|_{L^2}^2 &= \|u_1 \times \nabla^{k+1} B + [\nabla^{k+1}, u_1] \times B\|_{L^2}^2 \\ &\lesssim \|u_1 \times \nabla^{k+1} B\|_{L^2}^2 + \|[\nabla^{k+1}, u_1] \times B\|_{L^2}^2 \\ &\lesssim \|u_1\|_{L^\infty}^2 \|\nabla^{k+1} B\|_{L^2}^2 + \|\nabla u_1\|_{L^\infty}^2 \|\nabla^k B\|_{L^2}^2 + \|\nabla^{k+1} u_1\|_{L^2}^2 \|B\|_{L^\infty}^2. \end{aligned}$$

The other nonlinear terms on the right-hand side of (3.21) can be estimated similarly. Hence, we deduce from (3.21) that, by (1.11)–(1.13),

$$\begin{aligned} & \frac{d}{dt} \mathcal{G}_k(t) + \mathcal{G}_k(t) \\ & \lesssim \|\nabla^{k+2} n_2\|_{L^2}^2 + \|u_1\|_{L^\infty}^2 \|\nabla^{k+1} B\|_{L^2}^2 + \|\nabla u_1\|_{L^\infty}^2 \|\nabla^k B\|_{L^2}^2 + \|B\|_{L^\infty}^2 \|\nabla^{k+1} u_1\|_{L^2}^2 \\ & \quad + \|n_2\|_{L^\infty}^2 \|\nabla^k n_1\|_{L^2}^2 + \|(n_1, n_2, u_1, u_2)\|_{L^\infty}^2 \|\nabla^{k+2}(n_1, n_2, u_1, u_2)\|_{L^2}^2 \\ & \quad + \|\nabla(n_1, n_2, u_1, u_2)\|_{L^\infty}^2 \|\nabla^{k+1}(n_1, n_2, u_1, u_2)\|_{L^2}^2 \\ & \leq C_0 \left((1+t)^{-(k+3+s)} + (1+t)^{-(k+7/2+2s)} + (1+t)^{-(k+11/2+2s)} \right) \\ & \leq C_0(1+t)^{-(k+3+s)}, \end{aligned} \quad (3.22)$$

where we required $N \geq 2k + 8 + s$. Applying the Gronwall lemma to (3.22) again, we obtain

$$\mathcal{G}_k(t) \leq \mathcal{G}_k(0)e^{-t} + C_0 \int_0^t e^{-(t-\tau)} (1+\tau)^{-(k+3+s)} d\tau \leq C_0(1+t)^{-(k+3+s)}.$$

This implies

$$\|\nabla^k(n_2, \psi)(t)\|_{L^2} \lesssim \sqrt{\mathcal{G}_k(t)} \leq C_0(1+t)^{-\frac{k+3+s}{2}}. \quad (3.23)$$

If required $N \geq 2k + 12 + s$, then by (3.23), we have

$$\|\nabla^{k+2}n_2(t)\|_{L^2} \lesssim C_0(1+t)^{-\frac{k+5+s}{2}}.$$

Having obtained such faster decay, we can then improve (3.22) to be

$$\frac{d}{dt}\mathcal{G}_k(t) + \mathcal{G}_k(t) \leq C_0 \left((1+t)^{-(k+5+s)} + (1+t)^{-(k+7/2+2s)} \right) \leq C_0(1+t)^{-(k+7/2+2s)}.$$

Applying the Gronwall lemma again, we obtain

$$\|\nabla^k(n_2, \psi)(t)\|_{L^2} \lesssim \sqrt{\mathcal{G}_k(t)} \leq C_0(1+t)^{-(k/2+7/4+s)}.$$

We thus complete the proof of (1.14). The proof of Theorem 1.2 is completed.

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